Systems of singularly perturbed differential equations

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Model problem

Reaction-convection-diffusion:

$$-\boldsymbol{E}\boldsymbol{u}'' + \boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \ \text{ in } \ (0,1), \ \boldsymbol{u}(0) = \boldsymbol{u}(1) = \boldsymbol{0}$$

with

$$|\boldsymbol{E}| \ll \max\big\{|\boldsymbol{B}|, |\boldsymbol{A}|\big\}.$$

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Model problem

scalar convection-diffusion

$$-\varepsilon u'' + bu' + au = f \text{ in } (0,1),\ u(0) = u(1) = 0$$

scalar reaction-diffusion

$$-\varepsilon^2 u'' + au = f \text{ in } (0,1),\ u(0) = u(1) = 0$$

with

$$0 < \varepsilon \ll 1.$$

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Model problem

Convection-diffusion

$$-\varepsilon_1 u_1'' + b_{11}u_1' + \cdots + b_{1\ell}u_\ell' + a_{11}u_1 + \cdots + a_{1\ell}u_\ell = f_1$$

$$\vdots$$

$$-\varepsilon_\ell u_\ell'' + b_{\ell 1}u_1' + \cdots + b_{\ell\ell}u_\ell' + a_{\ell 1}u_1 + \cdots + a_{\ell\ell}u_\ell = f_\ell$$

short:

$$\boldsymbol{\mathcal{L}u} := -\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}'' + \boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$$

O'Riordan, Stynes $(\ell = 2)$	2009	AdCM
L.	2009	SINUM
Roos	2011	AML

Model problem

Reaction-diffusion

$$-\varepsilon_1^2 u_1'' + a_{11} u_1 + \cdots + a_{1\ell} u_\ell = f_1$$

$$\vdots$$

$$-\varepsilon_\ell^2 u_\ell'' + a_{\ell 1} u_1 + \cdots + a_{\ell\ell} u_\ell = f_\ell$$

short:

$$\boldsymbol{\mathcal{L}} \boldsymbol{u} := -\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$$

Madden, Stynes $(\ell = 2)$	... 2003	IMA J NA
L., Madden $(\ell = 2)$	... 2004	Computing
L., Madden	2003-2006	IMA J NA

Outline

- challenges of singularly perturbed problems
- What is a “singularly perturbed problem”?
- difficulties in their numerical treatment
- stability for scalar differential operators
  - maximum/comparison principles
  - stability, Green’s functions
  - error analysis
- systems of reaction-diffusion [convection-diffusion] eq’s
  - stability
  - error analysis

Difficulties

Character of differential equation changes when $\varepsilon \to 0$

- 2^{nd} order $\to$ 1^{st} order or algebraic
- elliptic $\to$ hyperbolic or algebraic

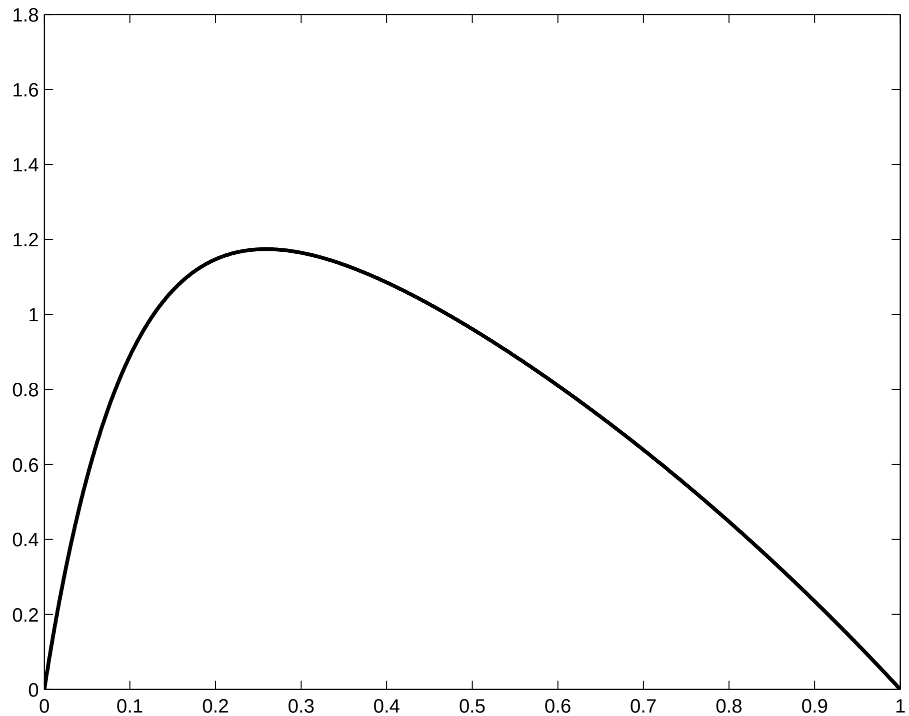
certain boundary conditions become superfluous

$$
\begin{aligned}
-\boldsymbol{E}\boldsymbol{u}'' + \boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} &= \boldsymbol{f} \\
\boldsymbol{B}\boldsymbol{u}' + \boldsymbol{A}\boldsymbol{u} &= \boldsymbol{f} \\
\boldsymbol{A}\boldsymbol{u} &= \boldsymbol{f}
\end{aligned}
$$

Difficulties

Example:

$$-\varepsilon u''(x) - u'(x) = e^x \text{ for } x \in (0,1),\; u(0) = u(1) = 0.$$



$\varepsilon = 10^{-1}$

Difficulties

Example:

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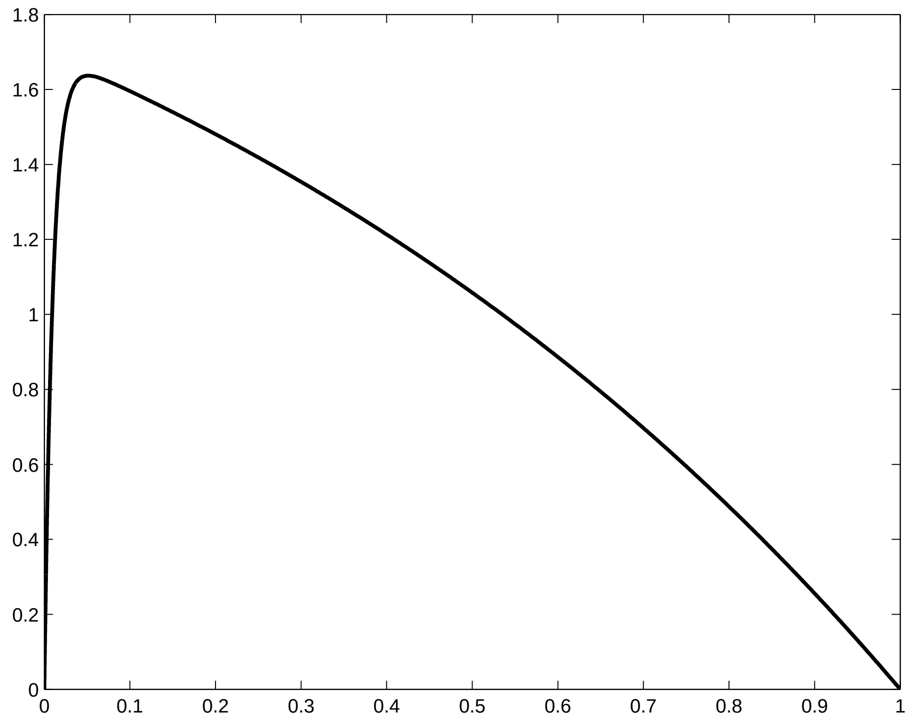
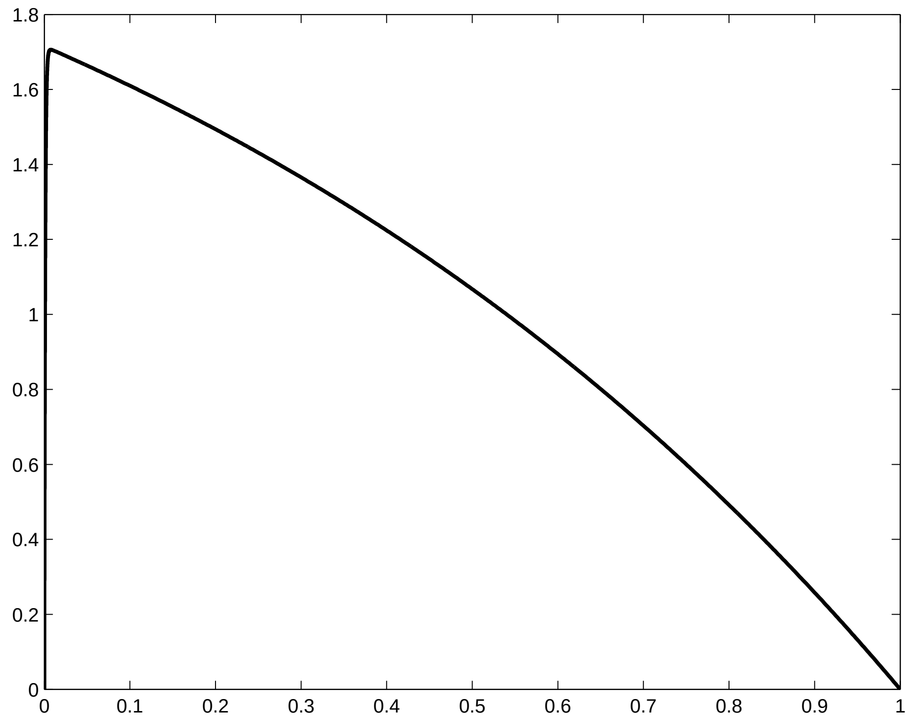


$\varepsilon = 10^{-2}$

Difficulties

Example:

$$-\varepsilon u''(x) - u'(x) = e^x \text{ for } x \in (0,1),\; u(0) = u(1) = 0.$$

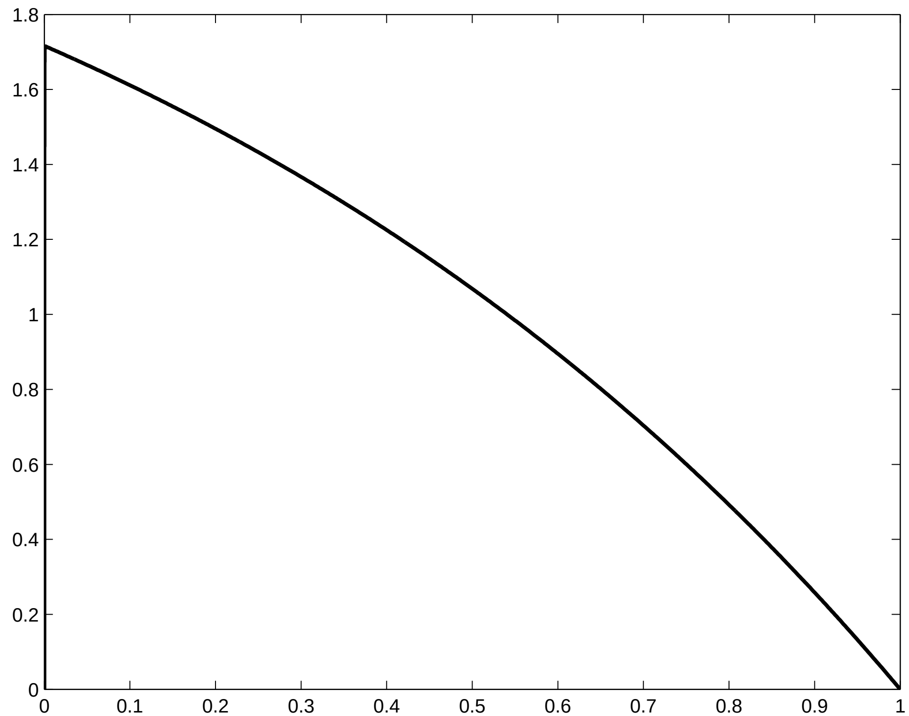


$\varepsilon = 10^{-3}$

Difficulties

Example:

$$-\varepsilon u''(x) - u'(x) = e^x \text{ for } x \in (0,1),\ u(0) = u(1) = 0.$$

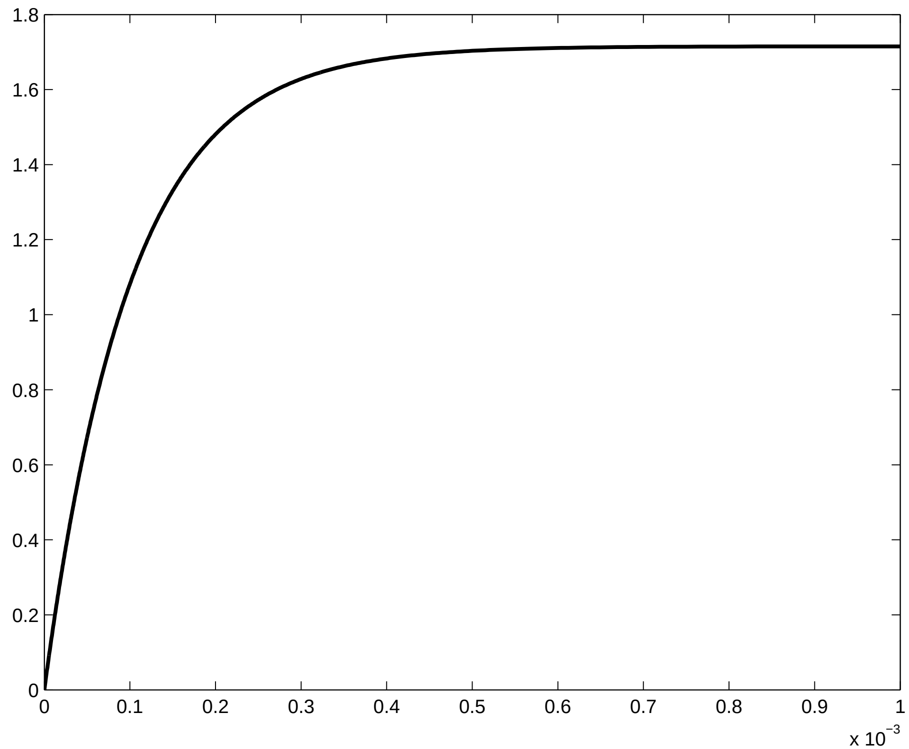


$\varepsilon = 10^{-4}$

Difficulties

Example:

$$-\varepsilon u''(x) - u'(x) = e^x \text{ for } x \in (0,1),\ u(0) = u(1) = 0.$$



$$\varepsilon = 10^{-4}$$

Difficulties

Discrepancy of

- prescribed boundary condition $u(0) = 0$ and
- solution u_{red} of reduced problem

$$-u'(x) = e^x \text{ for } x \in [0,1),\ u(1) = 0.$$

$u_{red}(0) = e - 1 \approx 1.72$

Consequence: solution changes rapidly near $x = 0$

$\Rightarrow$ boundary **layer** $\left|u^{(k)}(0)\right| \sim \varepsilon^{-k}$

$\varepsilon = 10^{-4}$: $|u'(0)| \approx 10^4$, $|u''(0)| \approx 10^8$ etc

Singularly perturbed problems

In example: $u : [0,1] \times (0,1] : (x,\varepsilon) \mapsto u(x,\varepsilon)$

$$\lim_{\varepsilon\to 0} \underbrace{\lim_{x\to 0} u(x,\varepsilon)}_{=0} = 0 \neq e - 1 = \lim_{x\to 0} \underbrace{\lim_{\varepsilon\to 0} u(x,\varepsilon)}_{=u_{red}(x)}.$$

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Singularly perturbed problems

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The solution of the boundary value problem possesses in $(x,\varepsilon) = (0,0)$ a “classical” singularity.

In **any** neighbourhood of $(x,\varepsilon) = (0,0)$ u attains **any** value between 0 and $e-1 \approx 1.72$.

Singularly perturbed problems

Definition.
Let E and B be two normed spaces. Let $D \subset E$ be an open subset. The continuous function $u : D \to B, \varepsilon \mapsto u(\varepsilon)$ is said to be regular for $\varepsilon \to \varepsilon^* \in \partial D$ if there exists a function $u^* \in B$ with

$$\boxed{\lim_{\varepsilon \to \varepsilon^*} \|u(\varepsilon) - u^*\|_B = 0;}$$

otherwise $u(\varepsilon)$ is said to be singular for $\varepsilon \to \varepsilon^*$.
A problem $(\mathcal{P}_\varepsilon)$ with solution $u(\varepsilon) \in B$, $\varepsilon \in D$, is said to be singularly perturbed for $\varepsilon \to \varepsilon^* \in \partial D$ in the **norm** $\|\cdot\|_B$, if u is singular for $\varepsilon \to \varepsilon^*$.

Here: $E = \mathbb{R}$, $D = (0,1]$, $B = C[0,1]$, $\varepsilon^* = 0$

Numerical approximation

Standard difference discretization.

mesh $\omega: \; x_i = ih, \; h = 1/N, \; u_i^N \approx u(x_i)$

$$-\frac{\varepsilon}{h}\left(\frac{u_{i+1}^N - u_i^N}{h} - \frac{u_i^N - u_{i-1}^N}{h}\right) - b(x_i)\frac{u_{i+1}^N - u_{i-1}^N}{2h} = f(x_i)$$

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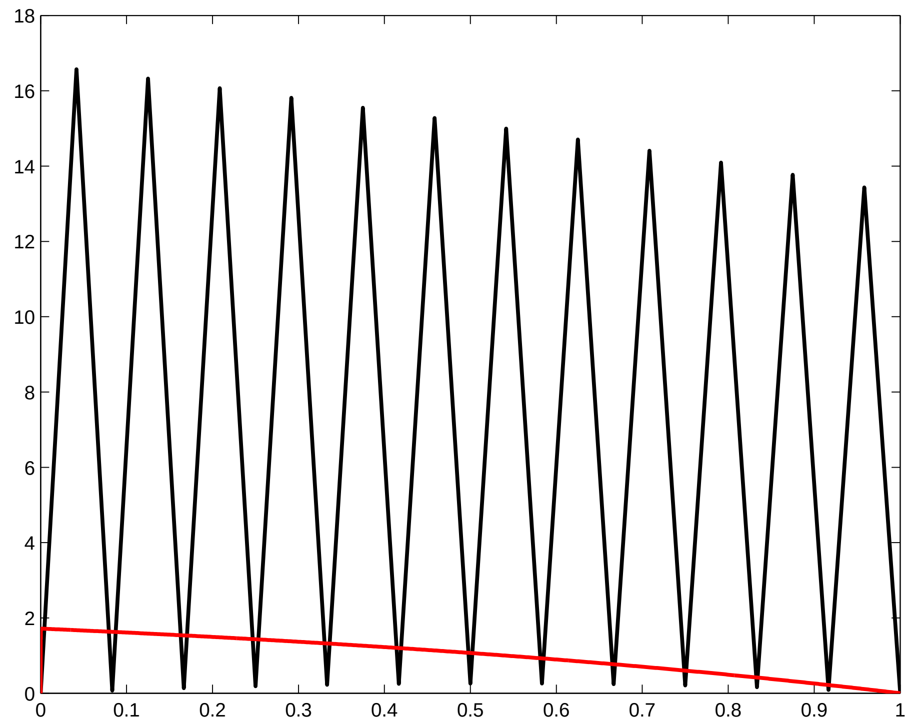

$$\varepsilon = 10^{-4}$$

Difference operator does **not** reflect stability properties of the differential operator.

Numerical approximation

Standard difference discretization.

mesh $\omega: \; x_i = ih, \; h = 1/N, \; u_i^N \approx u(x_i)$

$$-\frac{\varepsilon}{h}\left(\frac{u_{i+1}^N - u_i^N}{h} - \frac{u_i^N - u_{i-1}^N}{h}\right) - b(x_i)\frac{u_{i+1}^N - u_{i-1}^N}{2h} = f(x_i)$$

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Numerical approximation

Upwind difference scheme. (stabilized)

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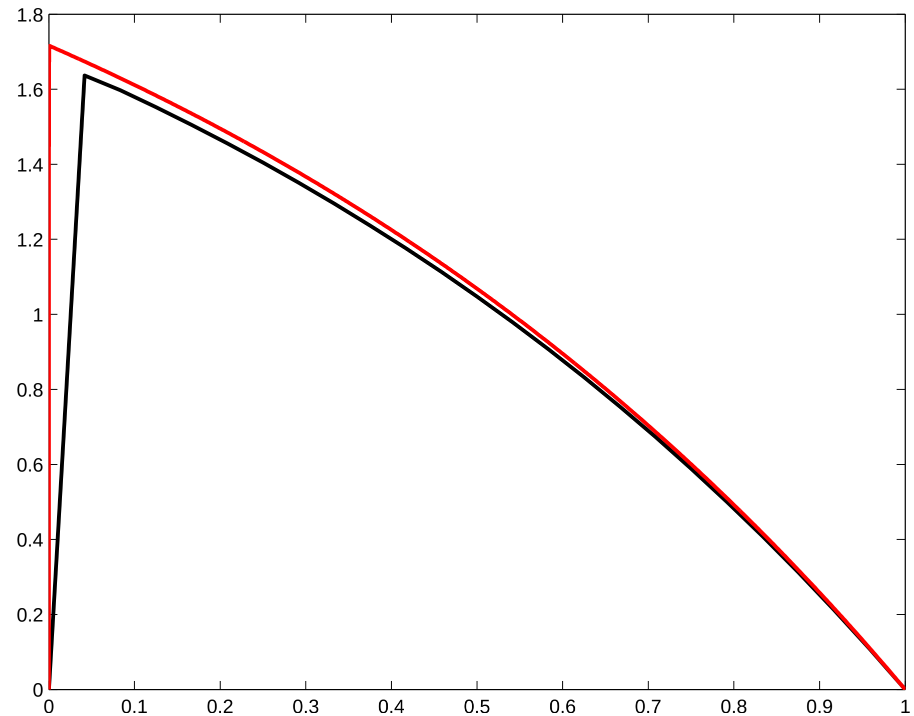

$\varepsilon = 10^{-4}$

Stability: :-)

Resolution of layer: $\varepsilon \sim h = N^{-1}$:-(

Numerical approximation

Typical error estimate:

$$\left\| u - u^N \right\| \le Kh, \quad \text{but} \quad K \sim \left\| u' \right\| \sim 1/\varepsilon$$

Good approximations only when $h \ll \varepsilon$, i.e., $N \gg \varepsilon^{-1}$

Numerical approximation

Typical error estimate:

$$\|u - u^N\| \leq Kh, \quad \text{but} \;\; K \sim \|u'\| \sim 1/\varepsilon$$

Good approximations only when $h \ll \varepsilon$, i.e., $N \gg \varepsilon^{-1}$

Two requirements

- Stability (upwind schemes)
- Resolution of layer (layer-resolving meshes)

Aim

- Applications: robust/uniform numerical methods

$$\left\|\boldsymbol{u}_{\varepsilon}-\boldsymbol{u}_{\varepsilon}^{N}\right\| \leq \vartheta(N)$$

where

N	number of mesh points
$\lim_{N\to\infty} \vartheta(N) = 0$	convergence
$\partial_{\varepsilon_k}\vartheta = 0$	robustness/uniformity
$\|\cdot\|$	a reasonable norm

- Mathematics: analytical properties
  - stability (of continuous and discrete operators)
  - layer structure (derivatives, dependence on ε)
  - nice, elegant and easy to communicate theory

Stability of differential operators

$$(\mathcal{L}v)(x) := -\mu v''(x) + b(x)v'(x) + a(x)v(x), \quad \mu > 0$$

Theorem. (Maximum principle)
Assume there exists a function $\psi \in C^2(0,1) \cap C[0,1]$ with

$$\psi > 0 \text{ on } [0,1] \quad \text{and} \quad \mathcal{L}\psi > 0 \text{ in } (0,1).$$

Then for $u \in C^2(0,1) \cap C[0,1]$

$$\left.\begin{aligned} \mathcal{L}u \geq 0, &\quad \text{in } (0,1) \\ u(0) \geq 0, & \\ u(1) \geq 0 & \end{aligned}\right\} \quad \Longrightarrow \quad u \geq 0, \quad \text{on } [0,1].$$

Proof. By contradiction, consider v defined by $\psi v = u$.

Stability of differential operators

Corollary. (Comparison principle)
Assume there exists a function $\psi \in C^2(0,1) \cap C[0,1]$ with

$$\psi > 0 \text{ on } [0,1] \quad \text{and} \quad \mathcal{L}\psi > 0 \text{ in } (0,1).$$

Then for any two functions $u, w \in C^2(0,1) \cap C[0,1]$

$$\left.\begin{array}{r} \mathcal{L}u \geq \mathcal{L}w, \quad \text{in } (0,1) \\ u(0) \geq w(0), \\ u(1) \geq w(1) \end{array}\right\} \implies \quad u \geq w, \quad \text{on } [0,1].$$

The operator $\mathcal{L}$ is said to be inverse monotone.

$\mathcal{L}u = f$ + bcs   possesses a unique solution

Stability of differential operators

Green's function. Given u with $u(0) = u(1) = 0$. Then

$$u(x) = \int_0^1 \mathcal{G}(x,\xi) \underbrace{\big(\mathcal{L}u\big)(\xi)}_{f(\xi)} d\xi$$

inverse monotonicity $\iff$ $\mathcal{G}(\cdot,\cdot) \geq 0$

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Stability of differential operators

Green's function. Given u with $u(0) = u(1) = 0$. Then

$$u(x) = \int_0^1 \mathcal{G}(x,\xi) \underbrace{(\mathcal{L}u)(\xi)}_{f(\xi)} d\xi$$

$$\text{inverse monotonicity} \quad \Longleftrightarrow \quad \mathcal{G}(\cdot,\cdot) \geq 0$$

Characterization of $\mathcal{G}$.

$$(\mathcal{L}\mathcal{G}(\cdot,\xi))(x) = \delta(x-\xi) \qquad (\mathcal{L}^*\mathcal{G}(x,\cdot))(\xi) = \delta(\xi - x)$$

Stability of differential operators

Green's function

$$u(x) = \int_0^1 \mathcal{G}(x,\xi)\big(\mathcal{L}u\big)(\xi)d\xi$$

Stability: Cauchy-Schwarz, Hölder, integration by parts

$$\|u\|_A \leq C\,\|\mathcal{L}u\|_B$$

For example

$$\begin{aligned}
\|\mathcal{G}(x,\cdot)\|_{L_1} \leq C \quad &\Longrightarrow \quad \|u\|_\infty \leq C\,\|\mathcal{L}u\|_\infty \\
\|\mathcal{G}(x,\cdot)\|_\infty \leq C \quad &\Longrightarrow \quad \|u\|_\infty \leq C\,\|\mathcal{L}u\|_{L_1} \\
\|\mathcal{G}(x,\cdot)\|_{W^{1,1}} \leq C \quad &\Longrightarrow \quad \|u\|_\infty \leq C\,\|\mathcal{L}u\|_{W^{-1,\infty}}
\end{aligned}$$

Stability of discrete operators

Discrete operators = matrices, $L = \left(l_{ij}\right) \in \mathbb{R}^{m,m}$

L is said to be inverse monotone if for any $u, w \in \mathbb{R}^m$

$$Lu \geq Lw \quad \Longrightarrow \quad u \geq w$$

Equivalent:

$$L^{-1} \geq 0.$$

Green's function $G = (g_{ij})$:

$$\boxed{u_i = \sum_{j=1}^{m} g_{ij}(Lu)_j} \qquad G = L^{-1}$$

Tool: M-matrix criterion

Error analysis

Differential equation

$$\mathcal{L}u = f \text{ in } (0,1), \quad u(0) = u(1) = 0.$$

mesh $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$, $\quad u_i^N \approx u(x_i)$

$$\left[L^N u^N\right]_i = f_i^N, \quad i = 1, \ldots, N-1, \; u_0^N = u_N^N = 0$$

A priori error estimate

$$L^N\left(u - u^N\right) = L^N u - f^N = \text{truncation error}$$

Stability of $L^N \Longrightarrow$

$$\boxed{\left\|u - u^N\right\|_{A,\omega} \leq C\left\|L^N u - f^N\right\|_{B,\omega}}$$

Error analysis

Differential equation

$$\mathcal{L}u = f \text{ in } (0,1), \quad u(0) = u(1) = 0.$$

mesh $\omega : 0 = x_0 < x_1 < \cdots < x_N = 1$, $\quad u_i^N \approx u(x_i)$

$$\left[L^N u^N\right]_i = f_i^N, \quad i = 1, \ldots, N-1, \; u_0^N = u_N^N = 0$$

A posteriori error estimate

$$\mathcal{L}\left(u - u^N\right) = f - \mathcal{L}u^N = \text{residuum}$$

Stability of $\mathcal{L} \Longrightarrow$

$$\boxed{\left\|u - u^N\right\|_A \leq C\left\|f - \mathcal{L}u^N\right\|_B}$$

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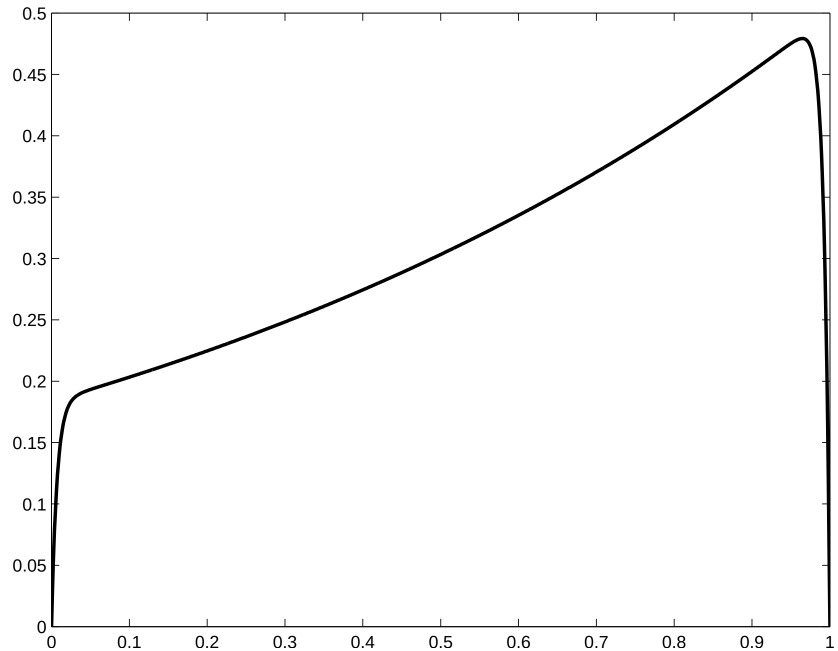
Reaction-diffusion $-\operatorname{diag}(\varepsilon^2)u'' + Au = f$

Scalar reaction-diffusion

$$-\varepsilon^2 u'' + au = f \text{ in } (0,1),\ u(0) = u(1) = 0,$$

$a(x) \geq \alpha^2$, $x \in [0,1]$, $\alpha > 0$.



$$-10^{-3}u''(x) + 2u(x) = e^{x-1}$$

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

Scalar reaction-diffusion

$$-\varepsilon^2 u'' + au = f \quad \text{in } (0,1),\ u(0) = u(1) = 0,$$

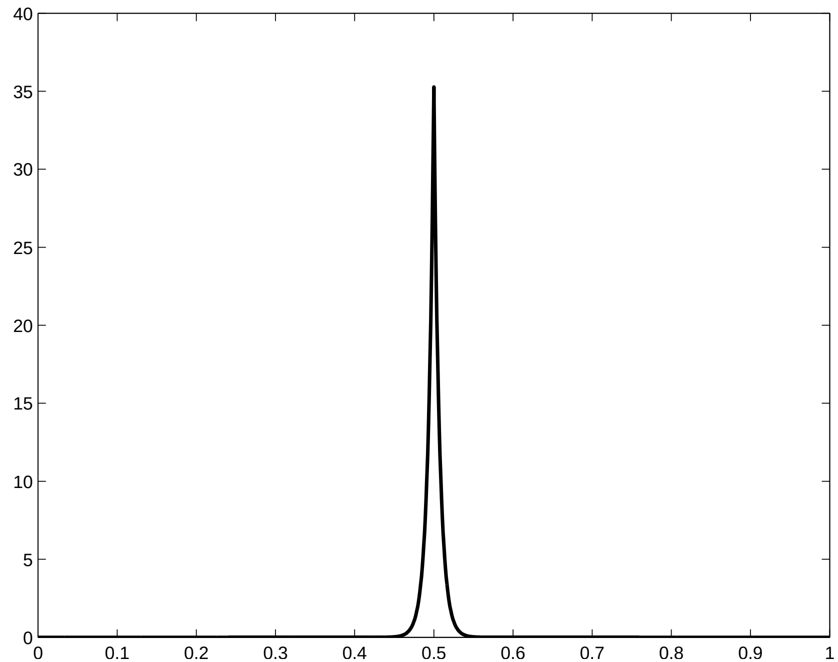
$a(x) \geq \alpha^2$, $x \in [0,1]$, $\alpha > 0$.

Derivative bounds

$$\left| u^{(k)}(x) \right| \leq C \left\{ 1 + \varepsilon^{-k} e^{-\alpha x/\varepsilon} + \varepsilon^{-k} e^{-\alpha(1-x)/\varepsilon} \right\}$$

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

Green's function: $\mathcal{G} := \mathcal{G}(x, \cdot)$



$$\mathcal{G} \geq 0, \qquad \int_0^1 a(\xi)\mathcal{G}(\xi)d\xi \leq 1, \qquad \int_0^1 \left|\mathcal{G}'(\xi)\right| d\xi \leq \frac{1}{\varepsilon\alpha},$$

$$\int_0^1 \left|\mathcal{G}''(\xi)\right| d\xi \leq \frac{2}{\varepsilon^2}.$$

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

Stability

$$u(x) = \int_0^1 a(\xi)\mathcal{G}(\xi)\frac{(\mathcal{L}u)(\xi)}{a(\xi)}d\xi$$

$$|u(x)| \leq \left\|\frac{\mathcal{L}u}{a}\right\|_\infty \underbrace{\int_0^1 a(\xi)\mathcal{G}(\xi)d\xi}_{\leq 1}$$

$$\boxed{\|u\|_\infty \leq \left\|\frac{\mathcal{L}u}{a}\right\|_\infty}$$

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

Systems. [L., Madden 2006]

- essential assumption: $a_{kk}(x) > 0$ for $x \in [0, 1]$
- for simplicity: two equations
- for simplicity: homogenious Dirichlet bc's

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

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Rewrite system

$$-\varepsilon_1^2 u_1'' + a_{11}u_1 = f_1 - a_{12}u_2$$
$$-\varepsilon_2^2 u_2'' + a_{22}u_2 = f_2 - a_{21}u_1$$

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

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…and use scalar stability!

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

Hence — with $\|\cdot\| := \|\cdot\|_\infty$ —

$$\|u_1\| \le \left\| \frac{f_1 - a_{12}u_2}{a_{11}} \right\| \le \left\| \frac{f_1}{a_{11}} \right\| + \left\| \frac{a_{12}}{a_{11}} \right\| \|u_2\|$$

$$\|u_2\| \le \left\| \frac{f_2 - a_{21}u_1}{a_{22}} \right\| \le \left\| \frac{f_2}{a_{22}} \right\| + \left\| \frac{a_{21}}{a_{22}} \right\| \|u_1\|$$

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

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Rearrange

$$\underbrace{\begin{pmatrix} 1 & -\|a_{12}/a_{11}\| \\ -\|a_{21}/a_{22}\| & 1 \end{pmatrix}}_{\text{inverse monotone?}} \begin{pmatrix} \|u_1\| \\ \|u_2\| \end{pmatrix} \le \begin{pmatrix} \|f_1/a_{11}\| \\ \|f_2/a_{22}\| \end{pmatrix}$$

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

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took me 3 years…

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

Theorem. Assume

- $a_{kk}(x) > 0$ for $x \in [0,1]$
- $$\Gamma := \begin{pmatrix} 1 & -\|a_{12}/a_{11}\| & \cdots & -\|a_{1\ell}/a_{11}\| \\ -\|a_{21}/a_{22}\| & 1 & \cdots & -\|a_{2\ell}/a_{22}\| \\ \vdots & \vdots & \ddots & \vdots \\ -\|a_{\ell 1}/a_{\ell\ell}\| & -\|a_{\ell 2}/a_{\ell\ell}\| & \cdots & 1 \end{pmatrix}$$

  is inverse monotone.

Then

$$\max_{k=1,\dots,\ell} \|u_k\| \le C \max_{k=1,\dots,\ell} \|f_k/a_{kk}\| .$$

…2D, 3D.

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

When is Γ inverse monotone?

- $\boldsymbol{A}$ is strongly diagonally dominant:

$$a_{kk}(x) > \sum_{i \neq k} |a_{ki}(x)| \quad \text{for } x \in [0,1]$$

  $\Longrightarrow$ M-criterion with $e \equiv 1$

- other cases..., for example

$$\boldsymbol{A} = \begin{pmatrix} 1 & -47.11 \\ 0 & 1 \end{pmatrix}$$

- Compute Γ exactly/approximately and check sign pattern of Γ^{-1} [and magnitude of entries].

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon^2})\boldsymbol{u''}+\boldsymbol{A}\boldsymbol{u}=\boldsymbol{f}$

Error analysis.

Discretization: central differencing

$$\boldsymbol{L}^N \boldsymbol{u}_i^N = -\operatorname{diag}\left(\boldsymbol{\varepsilon^2}\right)\delta^2 \boldsymbol{u}_i^N + \boldsymbol{A}(x_i)\boldsymbol{u}_i^N = \boldsymbol{f}(x_i) = \boldsymbol{f}^N \quad \text{on } \omega$$

scalar:

$$-\frac{2\varepsilon^2}{h_i + h_{i+1}}\left(\frac{u_{i+1}^N - u_i^N}{h_{i+1}} - \frac{u_i^N - u_{i-1}^N}{h_{i+1}}\right) + a(x_i)u_i^N = f(x_i)$$

Stability. [analogously to $\mathcal{L}$]

$$\max_{k=1,\ldots,\ell}\left\|u_k^N\right\|_\omega \le C \max_{k=1,\ldots,\ell}\left\|f_k/a_{kk}\right\|_\omega.$$

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

Error: $\boldsymbol{u} - \boldsymbol{u}^N$

Truncation error: $\boldsymbol{\tau} := \boldsymbol{L}^N\left(\boldsymbol{u} - \boldsymbol{u}^N\right) = \boldsymbol{L}^N\boldsymbol{u} - \boldsymbol{f}^N$

Decomposition: $\boldsymbol{\eta} = \boldsymbol{\psi} + \boldsymbol{\varphi}$

$$-\varepsilon_1^2\delta^2\psi_1 + a_{11}\psi_1 = \tau_1 = \varepsilon_1^2\left(u_1'' - \delta^2 u_1\right),$$

$$-\varepsilon_2^2\delta^2\psi_2 + a_{22}\psi_2 = \tau_2 = \varepsilon_2^2\left(u_2'' - \delta^2 u_2\right),$$

and

$$-\varepsilon_1^2\delta^2\varphi_1 + a_{11}\varphi_1 = -a_{12}\big(u_2 - u_2^N\big),$$

$$-\varepsilon_2^2\delta^2\varphi_2 + a_{22}\varphi_2 = -a_{21}\big(u_1 - u_1^N\big)$$

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

Scalar stability:

$$\|\varphi_1\|_\omega \le \left\|a_{12}/a_{11}\right\|_\omega \left\|u_2 - u_2^N\right\|_\omega$$
$$\|\varphi_2\|_\omega \le \left\|a_{21}/a_{22}\right\|_\omega \left\|u_1 - u_1^N\right\|_\omega$$

Triangle inequality

$$\left\|u_1 - u_1^N\right\|_\omega \le \|\psi_1\|_\omega + \|a_{12}/a_{11}\|_\omega \left\|u_2 - u_2^N\right\|_\omega$$
$$\left\|u_2 - u_2^N\right\|_\omega \le \|\psi_2\|_\omega + \|a_{21}/a_{22}\|_\omega \left\|u_1 - u_1^N\right\|_\omega$$

... imitate stability trick

$$\max_{k=1,\dots,\ell} \left\|u_k - u_k^N\right\|_\omega \le C \max_{k=1,\dots,\ell} \|\psi_k\|_\omega$$

Reaction-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon}^2)\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}$

The ψ_k are solutions of scalar equations. . .

Recycle known ideas for scalar problems:

- fancy barrier functions for the truncation error
- Green's function representations   :-) :-)
- or your favourite technique

in order to obtain a convergence result.

. . . provided we have bounds for the derivatives of the exact solution to estimate the truncation error!

Here: One pair of layers per equation: $\exp\left(-\alpha x/\varepsilon_k\right)$

. . . 2D, 3D?

Convection-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}'' + \boldsymbol{B}\boldsymbol{u}' = \boldsymbol{f}$

. . . with strong coupling

$$-\varepsilon_1 u_1'' + b_{11} u_1' + b_{12} u_2' = f_1$$
$$-\varepsilon_2 u_2'' + b_{21} u_1' + b_{22} u_2' = f_2$$

earlier idea:

$$-\varepsilon_1 u_1'' + b_{11} u_1' = f_1 - b_{12} u_2'$$

stability

$$\|u_1\| \le C \left\| f_1 - b_{12} u_2' \right\|$$ bad – does not fit

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Convection-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}''+\boldsymbol{B}\boldsymbol{u}'=\boldsymbol{f}$

Scalar problem:

$$-\mu u'' + bu' = f \quad \text{in } (0,1),\ u(0) = u(1) = 0$$

with $\quad b \geq \beta > 0 \quad$ or $\quad b \leq -\beta < 0 \quad$ in [0,1]

Then – standard stability

$$\|u\| \leq \beta^{-1}\,\|f\|$$

and – Andreev & Kopteva (1996), Andreev (2000)

$$\|u\| \leq 2\beta^{-1} \min_{F:F'=f} \|F\|$$

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Convection-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}'' + \boldsymbol{B}\boldsymbol{u}' = \boldsymbol{f}$

System: [L. 2009]

$$-\varepsilon_1 u_1'' + b_{11}u_1' + b_{12}u_2' = f_1$$
$$-\varepsilon_2 u_2'' + b_{22}u_2' + b_{21}u_1' = f_2$$

with $|b_{kk}(x)| \geq \beta_k > 0$ for $x \in [0,1]$.

$$-\varepsilon_1 u_1'' + b_{11}u_1' = f_1 - b_{12}u_2'$$
$$-\varepsilon_2 u_2'' + b_{22}u_2' = f_2 - b_{21}u_1'$$

…use scalar stability!

Convection-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}''+\boldsymbol{B}\boldsymbol{u}'=\boldsymbol{f}$

$$\|u_1\| \le \beta_1^{-1}\|f_1\| + 2\beta_1^{-1}\|b_{12}\|\,\|u_2\|$$
$$\|u_2\| \le \beta_2^{-1}\|f_2\| + 2\beta_2^{-1}\|b_{21}\|\,\|u_1\|$$

Rearrange

$$\underbrace{\begin{pmatrix} 1 & -2\beta_1^{-1}\|b_{12}\| \\ -2\beta_2^{-1}\|b_{21}\| & 1 \end{pmatrix}}_{\text{inverse monotone?}} \begin{pmatrix} \|u_1\| \\ \|u_2\| \end{pmatrix} \le \begin{pmatrix} \beta_1^{-1}\|f_1\| \\ \beta_2^{-1}\|f_2\| \end{pmatrix}$$

Convection-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}'' + \boldsymbol{B}\boldsymbol{u}' = \boldsymbol{f}$

Theorem. Assume

- $|b_{kk}(x)| \geq \beta_k > 0$ for $x \in [0,1]$
- $$\Gamma := \begin{pmatrix} 1 & -2\,\|b_{12}\|/\beta_1 & \cdots & -2\,\|b_{1\ell}\|/\beta_1 \\ -2\,\|b_{21}\|/\beta_2 & 1 & \cdots & -2\,\|b_{2\ell}\|/\beta_2 \\ \vdots & \vdots & \ddots & \vdots \\ -2\,\|b_{\ell 1}\|/\beta_\ell & -2\,\|b_{\ell 2}\|/\beta_\ell & \cdots & 1 \end{pmatrix}$$

  is inverse monotone.

Then

$$\max_{k=1,\ldots,\ell} \|u_k\| \leq C \max_{k=1,\ldots,\ell} \|f_k\| \,.$$

Convection-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}'' + \boldsymbol{B}\boldsymbol{u}' = \boldsymbol{f}$

Error analysis for a stabilized formally first order method:

- decomposition of error
- imitate stability analysis
- reduce to scalar problems

Results:

$$\max_{k=1,\dots,\ell} \left\| u_k - u_k^N \right\|_\omega \le C \max_{j=1,\dots,N} \int_{x_{j-1}}^{x_j} \left\{ 1 + \sum_{m=1}^{\ell} \left| u_m'(s) \right| \right\} ds$$

$$\max_{k=1,\dots,\ell} \left\| u_k - u_k^N \right\|_\omega \le C \max_{j=1,\dots,N} h_j \left\{ 1 + \sum_{m=1}^{\ell} \frac{\left| u_{m,j}^N - u_{m,j-1}^N \right|}{h_j} \right\} ds$$

Convection-diffusion $-\operatorname{diag}(\boldsymbol{\varepsilon})\boldsymbol{u}''+\boldsymbol{B}\boldsymbol{u}'=\boldsymbol{f}$

Open issues:

- derivative bounds
  - L_1-bounds [L. 2009]

$$\int_0^1 \left|u_m'(s)\right| ds \le C$$

$\Rightarrow$ existence of an optimal mesh

  - pointwise bounds: first guess [L. 2009]

$$e^{-\beta_m x/\varepsilon} \quad \text{if} \quad b_{m,m} \ge \beta_m > 0, \quad \ldots \text{wrong}$$

  - pointwise bounds: eigenvalues of $\boldsymbol{B}$ [Roos 2011]
- 2D, 3D: available stability results not strong enough

Conclusions

A simple trick allows to extend stability results for scalar problems to systems of singularly perturbed equations.

Solved problems.

- stability for reaction-diffusion and 1D convection-diffusion
- derivative bounds for reac-diff and conv-diff in 1D (including a priori and a posteriori error bounds)

Conclusions

A simple trick allows to extend stability results for scalar problems to systems of singularly perturbed equations.

Open issues.

- derivative bounds for higher-dimensional problems (corners!!)
- strong stability inequalities for higher dimensional convection-diffusion

Conclusions

A simple trick allows to extend stability results for scalar problems to systems of singularly perturbed equations.

Survey paper.

- Linß & Stynes: *Numerical solution of systems of singularly perturbed differential equations*, Comput. Meth. Appl. Math., 2009

Conclusions

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. . . The End. Thank you!