Adaptive Approximations for PDE-Constrained Parabolic Control Problems

Angela Kunoth
Universität Paderborn, Germany

Talk I: Deterministic control problems
M. Gunzburger, A. Kunoth,
Space-time adaptive wavelet methods for control problems constrained by parabolic evolution equations, SIAM J. Contr. Optim. **49**(3) (2011), 1150–1170.

Talk II: Control problems with stochastic coefficients
A. Kunoth and Ch. Schwab,
Analytic regularity and gpc approximation for control problems constrained by linear parametric elliptic and parabolic pdes, SAM preprint #2011-54, ETH Zürich, revised, October 2012, in revision.

A. Kunoth and Ch. Schwab,
Sparse adaptive tensor Galerkin approximations of stochastic PDE-constrained control problems, Manuscript, February 2013.

Supported by the EU's 7th Framework Programme (FP7-REGPOT-2009-1)
Grant Agreement Nr. 245749

Optimization Problems: First Order Necessary Conditions

Constrained minimization problem

$\inf_{(y,u)\in Y\times U}$	$J(y,u)$	$J: Y \times U \to \mathbb{R}$	Y, U reflexive Banach spaces
subject to	$K(y,u) = 0$	$K: Y \times U \to Y'$	

Assumption: for given control $u \in U$, there exists a unique state $y \in Y$

Solution approach: compute zeroes of first order Fréchet derivatives of Lagrangian functional

$L(y,u,p) := J(y,u) + \langle K(y,u), p\rangle_{Y' \times Y}$ $\qquad L: Y \times U \times Y \to \mathbb{R}$ $\qquad$ costate/adjoint p

$$\rightsquigarrow \qquad \delta L(y,u,p) := \begin{pmatrix} L_y(y,u,p) \\ L_u(y,u,p) \\ L_p(z,u,p) \end{pmatrix} = 0 \qquad \Longleftrightarrow \qquad \begin{pmatrix} J_y(y,u) + \langle K_y(y,u), p\rangle_{Y' \times Y} \\ J_u(y,u) + \langle K_u(y,u), p\rangle_{Y' \times Y} \\ K(y,u) \end{pmatrix} = 0$$

Special case: J quadratic in y, u, K linear in y, u

$\rightsquigarrow$ linear system (Karush-Kuhn-Tucker (KKT) system)

$$\begin{pmatrix} L_{yy} & L_{yu} & K_y^* \\ L_{uy} & L_{uu} & K_u^* \\ K_y & K_u & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \end{pmatrix} = g \quad \Longleftrightarrow: \quad \begin{pmatrix} \mathcal{A} & \mathcal{B}^* \\ \mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} (y,u)^T \\ p \end{pmatrix} = g \quad \Longleftrightarrow: \quad G\,q = g$$

$\langle C^* q, r\rangle := \langle q, Cr\rangle$ $\qquad$ and necessary conditions are also sufficient

Optimal Control Problem Constrained by a Parabolic PDE with Distributed Control

Given $y_*(t)$ f $\omega > 0$ end time $T > 0$ initial condition y_0

$$
\begin{array}{lrcll}
\text{minimize} & J(y,u) & = & \frac{1}{2}\int_0^T \|y(t,\cdot) - y_*(t,\cdot)\|_Z^2 \, dt + \frac{\omega}{2}\int_0^T \|u(t,\cdot)\|_U^2 \, dt & \\
\text{subject to} & y'(t) + A(t)y(t) & = & f(t) + u(t) \qquad \text{a.e. } t \in (0,T) =: I & \text{(PDE)} \\
 & y(0) & = & y_0 &
\end{array}
$$

$y' := \frac{\partial}{\partial t} y$ $y = y(t,x)$ state $u = u(t,x)$ control

$V = H_0^1(\Omega)$ state space $Z = H_0^1(\Omega)$ observation space $U = H^{-1}(\Omega)$ control space

$A(t) : V \to V'$ $\langle A(t)v(t,\cdot), w(t,\cdot)\rangle := \int_\Omega [\nabla v(t,x)\cdot\nabla w(t,x) + v(t,x)w(t,x)]\, dx$ $\Omega \subset \mathbb{R}^d$

$A(t)$ 2nd order linear selfadjoint coercive & continuous operator on V

PDE-constrained control problem $\rightsquigarrow$ requires repeated solution of PDE constraint

$$
\begin{aligned}
y'(t) + A(t)y(t) &= f(t) + u(t) \\
y(0) &= y_0
\end{aligned}
$$

Necessary and Sufficient Conditions for Optimality

Optimal control problem constrained by a parabolic PDE

$\rightsquigarrow$ system of parabolic PDEs coupled globally in time

$$
\begin{aligned}
y'(t) + A(t)\,y(t) &= f(t) + u(t) && \text{a.e. } t \in I\\
y(0) &= y_0 &&\\
-p'(t) + A(t)^T p(t) &= -R\,(y(t) - y_*(t)) && \text{a.e. } t \in I\\
p(T) &= 0 &&\\
\omega\, R^{-1} u(t) + p(t) &= 0 && \text{a.e. } t \in I
\end{aligned}
$$

Riesz operator R defined by $\langle Rv, v\rangle := \|v\|^2_{L_2(I)\otimes V}$

Obstructions for numerical solution:

- convential time discretizations: time-marching methods
  $\rightsquigarrow$ need storage of $y(t_i), p(t_i), u(t_i)$ for all discrete times $0 = t_0, \dots, T = t_N$
- in each time step: solve elliptic PDE $\rightsquigarrow$ large linear system of equations
  $\rightsquigarrow$ iterative solver $\rightsquigarrow$ need preconditioning in (conjugate) gradient method
- singularities in data/domain: adaptive (FE) mesh(es) for $y(t_i), p(t_i), u(t_i)$ for all t_i
  one mesh for all variables ? refinement/coarsening ?
  convergence ? complexity ??
- adaptive space-time discretizations for control problems: one grid [Oeltz '06], [Meidner, Vexler '07], ...

Solution Ansatz here: full weak space-time form of parabolic PDE constraint

Variational Space-Time Form for a Single Parabolic Evolution PDE

[Dautray, Lions '92], [Schwab, Stevenson '09]

(PDE)
$$\begin{aligned} y'(t) + A(t)\,y(t) &= f(t) \qquad \text{a.e. } t \in I \\ y(0) &= y_0 \end{aligned}$$

solution space: Lebesgue-Bochner space $X := (L_2(I) \otimes V) \cap (H^1(I) \otimes V') \hookrightarrow \mathcal{C}^0(\bar{I}) \otimes L_2(\Omega)$

with norm $\|w\|_X^2 := \|w\|_{L_2(I)\otimes V}^2 + \|w'\|_{H^1(I)\otimes V'}^2$

test space: $Y := (L_2(I) \otimes V) \times L_2(\Omega)$ with norm $\|v\|_Y^2 := \|v_1\|_{L_2(I)\otimes V}^2 + \|v_2\|_{L_2(\Omega)}^2$

bilinear form $b(\cdot,\cdot) : X \times Y \to \mathbb{R}$

$$b(y,(v_1,v_2)) := \int_I [\langle y'(t,\cdot), v_1(t,\cdot)\rangle + \langle A(t)y(t,\cdot), v_1(t,\cdot)\rangle]\; dt + \langle y(0,\cdot), v_2\rangle =: \langle By, v\rangle$$

right hand side

$$\langle f, v\rangle := \int_I \langle f(t,\cdot), v_1(t,\cdot)\rangle\, dt + \langle y_0, v_2\rangle$$

(PDE) $\rightsquigarrow$ given $f \in Y'$, find $y \in X$: $By = f$

Theorem $\|Bw\|_{Y'} \sim \|w\|_X$ for all $w \in X$ mapping property (MP)

Reformulation of PDE-Constrained Optimal Control Problem

minimize	$J(y,u) = \frac{1}{2}\|y - y_*\|^2_{L_2(I)\otimes V} + \frac{\omega}{2}\|u\|^2_{L_2(I)\otimes V'}$	
subject to	$B\,y = f + u$ (PDE)	$B: X \to Y'$ satisfies (MP)

Necessary and Sufficient Conditions

$$L(y,u,p) := J(y,u) + \langle p,\ By - (f+u)\rangle$$

$$\delta L = 0 \ \rightsquigarrow\ \begin{array}{rcl} By &=& f + u \\ B^* p &=& R(y_* - y) \\ \omega R^{-1} u &=& p \end{array}$$

$$\Longleftrightarrow\ \begin{pmatrix} R & B^* \\ B & -\frac{1}{\omega}R \end{pmatrix}\begin{pmatrix} y \\ p \end{pmatrix} = \begin{pmatrix} Ry_* \\ f \end{pmatrix} \quad \text{(SPP)}$$

$\rightsquigarrow$ saddle point operator $\langle Gq, \tilde{q}\rangle := \left\langle \begin{pmatrix} R & B^* \\ B & -\frac{1}{\omega}R \end{pmatrix} q, \tilde{q} \right\rangle$

symmetric, continuous, boundedly invertible on $X \times X$

$\Longrightarrow$ unique solution $\begin{pmatrix} y \\ p \end{pmatrix}$ of system of PDEs (SPP)

Next: discretization in space and time variables by adaptive wavelet schemes

Building Blocks: (Biorthogonal Spline–) Wavelets

H Hilbert space on domain $\Omega \subset \mathbb{R}^d$ with $\|\cdot\|_H$ $\qquad$ H' dual space for H with $\langle \cdot, \cdot \rangle$

$\Psi := \{\psi_\lambda : \lambda \in \mathbb{I}\} \subset H$ **Wavelets** $\qquad$ $\mathbb{I}$ (infinite) index set

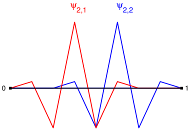
(NE) Ψ Riesz basis for H

$$v \in H: \quad v = \mathbf{v}^T \Psi := \sum_{\lambda \in \mathbb{I}} \langle v, \tilde{\psi}_\lambda \rangle \, \psi_\lambda \qquad \text{such that} \quad \|v\|_H \sim \|\mathbf{v}\|_{\ell_2(\mathbb{I})}$$

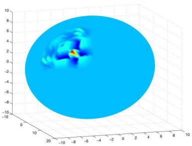
(L) Locality $\qquad \operatorname{diam}(\operatorname{supp} \psi_\lambda) \sim 2^{-|\lambda|}$ $\qquad$ $|\lambda|$ resolution

ψ_λ centered around $2^{-|\lambda|} k$

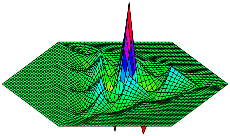
(CP) Vanishing moments $\langle v, \psi_\lambda \rangle \lesssim 2^{-|\lambda|(\frac{d}{2}+\tilde{m})} \|v^{(\tilde{m})}\|_{L_\infty(\operatorname{supp} \psi_\lambda)}$ for some $\tilde{m}$



[Dahmen, Kunoth, Urban '99]



[Dahmen, Schneider '99], [Kunoth, Sahner '06]



[Harbrecht, Schneider '00]

Paradigm of Adaptive Wavelet Method for One Stationary PDE

[Cohen, Dahmen, DeVore '99–'01]

(i) Well–posed variational problem: given $f \in Y'$, $B : X \to Y'$, find $v \in X$ such that $Bv = f$

(MP) $\|Bw\|_{Y'} \sim \|w\|_X$ for all $w \in X$ mapping property

(ii) Ψ^X, Ψ^Y wavelet bases for X, Y :

(NE) $\|\mathbf{w}^T \Psi^X\|_X \sim \|\mathbf{w}\|_{\ell_2}$ for all $\mathbf{w} = (w_\lambda)_{\lambda \in \mathbb{I}} \in \ell_2$

$\mathbf{Bv} := (\langle \psi_\lambda^Y, Bv \rangle)_{\lambda \in \mathbb{I}} \quad \mathbf{f} = (\langle \psi_\lambda^Y, f \rangle)_{\lambda \in \mathbb{I}}$

$\rightsquigarrow$ **Theorem** $Bv = f \iff \mathbf{Bv} = \mathbf{f}$ $\quad \mathbf{B} : \ell_2 \to \ell_2$ and $\mathbf{Bv} = \mathbf{f}$ well-posed in ℓ_2

(MP) + (NE) $\Longrightarrow$ $\|\mathbf{Bw}\|_{\ell_2} \sim \|\mathbf{w}\|_{\ell_2}$ for all $\mathbf{w} \in \ell_2$

(iii) (Idealized) iteration (for symmetric $\mathbf{B}$)
$\mathbf{v}^{n+1} = \mathbf{v}^n + (\mathbf{f} - \mathbf{Bv}^n) \quad n = 0, 1, 2, \ldots \qquad \|\mathbf{v}^{n+1} - \mathbf{v}\|_{\ell_2} \leq \rho \|\mathbf{v}^n - \mathbf{v}\|_{\ell_2} \quad \rho < 1$

(iv) Approximate realization through adaptive evaluation of $\mathbf{Bv}^n$ in routine SOLVE $[\varepsilon, \mathbf{B}, \mathbf{f}] \to \mathbf{v}_\varepsilon$

Extension to a Single Parabolic Evolution PDE

[Schwab, Stevenson '09]

(i) Variational space-time form of (PDE)

$$\begin{aligned} y'(t) + A(t)\,y(t) &= f(t) \qquad \text{a.e. } t \in I \\ y(0) &= y_0 \end{aligned}$$

solution space: Lebesgue-Bochner space $X := (L_2(I) \otimes V) \cap (H^1(I) \otimes V')$

with norm $\|w\|_X^2 := \|w\|_{L_2(I)\otimes V}^2 + \|w'\|_{H^1(I)\otimes V'}^2$

test space $Y := L_2(I; V) \times L_2(\Omega)$ with norm $\|v\|_Y^2 := \|v_1\|_{L_2(I)\otimes V}^2 + \|v_2\|_{L_2(\Omega)}^2$

bilinear form $b(\cdot,\cdot) : X \times Y \to \mathbb{R}$

$$b(y,(v_1,v_2)) := \int_I \left[\langle y'(t,\cdot), v_1(t,\cdot)\rangle + \langle A(t)y(t,\cdot), v_1(t,\cdot)\rangle\right] dt + \langle y(0,\cdot), v_2\rangle =: \langle By, v\rangle$$

right hand side

$$\langle f, v\rangle := \int_I \langle f(t,\cdot), v_1(t,\cdot)\rangle\, dt + \langle y_0, v_2\rangle$$

(PDE) $\rightsquigarrow$ given $f \in Y'$, find $y \in X$: $\boxed{By = f}$ [Dautray, Lions '92]

Theorem	(MP)	$\|Bw\|_{Y'} \sim \|w\|_X$	for all $w \in X$	mapping property

(ii) Ψ^X, Ψ^Y wavelet bases for X, Y $\rightsquigarrow$ $\mathbf{By} := (\langle \psi_\lambda^Y, By\rangle)_{\lambda\in\mathbb{I}}$ $\quad \mathbf{f} = (\langle \psi_\lambda^Y, f\rangle)_{\lambda\in\mathbb{I}}$

Theorem $By = f \iff \mathbf{By} = \mathbf{f}$ $\quad \mathbf{B} : \ell_2 \to \ell_2$ and $\mathbf{By} = \mathbf{f}$ well-posed in ℓ_2

(MP) + (NE) $\Longrightarrow$ $\|\mathbf{Bv}\|_{\ell_2} \sim \|\mathbf{v}\|_{\ell_2}$, $\mathbf{v} \in \ell_2$ $\qquad$ **B** unsymmetric

Application to PDE-Constrained Optimal Control Problem

Control problem in wavelet coordinates

minimize $\quad \mathbf{J}(\mathbf{y}, \mathbf{u}) = \frac{1}{2} \|\mathbf{y} - \mathbf{y}_*\|^2 + \frac{\omega}{2} \|\mathbf{u}\|^2$

subject to $\quad \mathbf{B}\mathbf{y} = \mathbf{f} + \mathbf{u}$

$\mathbf{B} : \ell_2 \to \ell_2$ automorphism $\qquad \|\cdot\| := \|\cdot\|_{\ell_2}$

Necessary and Sufficient Conditions

$$\mathbf{L}(\mathbf{y}, \mathbf{u}, \mathbf{p}) := \mathbf{J}(\mathbf{y}, \mathbf{u}) + \langle \mathbf{p}, \mathbf{B}\mathbf{y} - (\mathbf{f} + \mathbf{u}) \rangle$$

$$\delta \mathbf{L} = 0 \rightsquigarrow \boxed{\begin{array}{rcl} \mathbf{B}\mathbf{y} & = & \mathbf{f} + \mathbf{u} \\ \mathbf{B}^T \mathbf{p} & = & -(\mathbf{y} - \mathbf{y}_*) \\ \omega \mathbf{u} & = & \mathbf{p} \end{array}} \iff \boxed{\mathbf{Q}\mathbf{u} = \mathbf{g}}$$

$\mathbf{Q} : \ell_2 \to \ell_2$ automorphism

where
$$\mathbf{Q} := \mathbf{B}^{-T}\mathbf{B}^{-1} + \omega \mathbf{I}$$
$$\mathbf{g} := \mathbf{B}^{-T}(\mathbf{y}_* - \mathbf{B}^{-1}\mathbf{f})$$

Complexity Analysis

Based on benchmark:
decay rate s for (wavelet-)best N term approximation $\mathcal{A}^s := \{\mathbf{v} \in \ell_2 : \|\mathbf{v} - \mathbf{v}_N\| \lesssim N^{-s}\}$

Work/accuracy balance of best N term approximation:

Target accuracy ε $(\sim N^{-s})$ $\longleftrightarrow$ Work $\varepsilon^{-1/s}$ $(\sim N)$

Convergence and Complexity

(Idealized) iteration (for symmetric $\mathbf{B}$)
$\mathbf{v}^{n+1} = \mathbf{v}^n + (\mathbf{f} - \mathbf{B}\mathbf{v}^n)$ update via $\textsc{Res}\,[\eta, \mathbf{B}, \mathbf{f}, \mathbf{v}] \to \mathbf{r}_\eta$ $\rightsquigarrow$ $\textsc{Solve}\,[\varepsilon, \mathbf{B}, \mathbf{f}] \to \mathbf{v}_\varepsilon$

Theorem [Cohen, Dahmen, DeVore '99-'01]

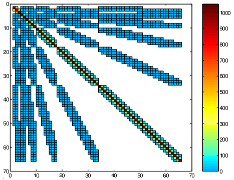
Vanishing moments (CP) for wavelets $\Longrightarrow$ $\mathbf{B}$ is s^*–compressible

$\Longrightarrow$ for variational problem satisfying (MP) scheme $\textsc{Solve}$ can be designed with properties:

(I) For every target accuracy $\varepsilon > 0$ $\textsc{Solve}$ produces after finitely many steps approximate solution $\mathbf{v}_\varepsilon$ such that $\|\mathbf{v} - \mathbf{v}_\varepsilon\| \leq \varepsilon$

(II) Exact solution $\mathbf{v} \in \mathcal{A}^s$ $\Longrightarrow$ $\operatorname{supp} \mathbf{v}_\varepsilon$, # flops $\sim \varepsilon^{-1/s} \sim N$

Core Ingredient of SOLVE : Compressible Operators



(CP) $\rightsquigarrow$ $\mathbf{B}$ is s^*–compressible:
for every $0 < s < s^*$ there exists $\mathbf{B}_j$ with $\leq \alpha_j 2^j$ nonzero entries per row and column such that

$$\|\mathbf{B} - \mathbf{B}_j\| \leq \alpha_j 2^{-sj} \; j \in \mathbb{N}_0 \; \sum_{j \in \mathbb{N}_0} \alpha_j < \infty$$ ($\mathbf{B}$ 'close to' sparse matrix)

Application of (Non)Linear Operators in Wavelet Bases

Theory [Cohen, Dahmen, DeVore '03] $d = 2$ [Vorloeper '10] general d, isotropic tensor product wavelets [Mollet, Pabel '12]

Input: finitely supported vector $\mathbf{v} = (v_\mu)_{\mu \in \Lambda}$ $\Lambda \subset \mathbb{I}$ finite
Output: approximation of $\mathbf{Bv}$ with infinite-dimensional operator $\mathbf{B} : \ell_2(\mathbb{I}) \to \ell_2(\mathbb{I})$
$B : X \to Y' \rightsquigarrow$ expand $Bv \in Y'$ in dual wavelet basis for Y and v in primal wavelet basis for X:

$$Bv = (\mathbf{Bv})^T \tilde{\Psi} = \sum_{\lambda \in \mathbb{I}} \langle Bv, \psi_\lambda \rangle \, \tilde{\psi}_\lambda = \sum_{\lambda \in \mathbb{I}} \langle B(\sum_{\mu \in \Lambda} v_\mu \psi_\mu, \psi_\lambda \rangle \, \tilde{\psi}_\lambda = \sum_{\lambda \in \mathbb{I}} \sum_{\mu \in \Lambda} v_\mu \langle B\psi_\mu, \psi_\lambda \rangle \, \tilde{\psi}_\lambda$$

$\rightsquigarrow$ compute $\langle B\psi_\mu, \psi_\lambda \rangle$ for given $\mu \in \Lambda$ (finite) and all $\lambda \in \mathbb{I}$

Compressibility of B: $$|\langle B\psi_\mu, \psi_\lambda \rangle| \leq C_{\|\mathbf{v}\|} \sup_{\mu:\, S_\lambda \cap S_\mu \neq \emptyset} 2^{-\gamma(|\lambda| - |\mu|)} \, |v_\mu| \qquad \gamma > \frac{d}{2} + 1$$

follows from wavelet property (CP)

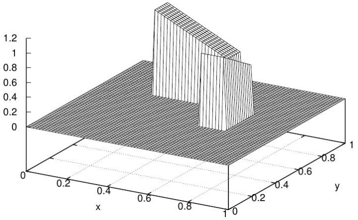
Essential data structure (for nonlinear operators): tree-type index sets
input $\mathbf{v} \rightsquigarrow$ prediction of tree index set based on supp $\mathbf{v}$ and properties of $\mathbf{B}$
$\rightsquigarrow$ computation of $(\mathbf{Bv})_\lambda$ after transformation to piecewise polynomials
$\rightsquigarrow$ application of $\mathbf{B}$ in optimal linear complexity

Application of (Non)Linear Operators in Wavelet Bases: Numerical Example

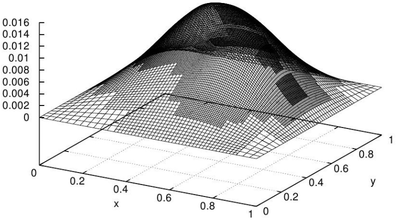
[Mollet, Pabel '12]

PDE with nonlinear term

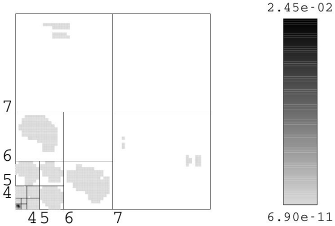
$$\begin{aligned} -\Delta y + y^3 &= f && \text{in } \Omega := (0,1)^2 \\ y &= 0 && \text{on } \partial\Omega \end{aligned}$$



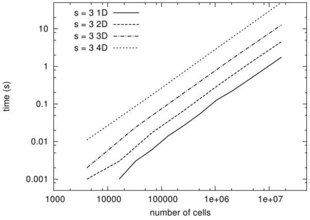
right hand side f



solution y (with Richardson scheme and residual error bound 10^{-3})



distribution of 7177 active wavelet coefficients



Runtime (seconds) for evaluating y^3 for $d \leq 4$

Convergence and Complexity Analysis for Control Problem with Parabolic PDE Constraints

Essential idea: RES for SOLVE $[\ldots, \mathbf{Q}, \ldots]$ reduced to RES for SOLVE $[\ldots, \mathbf{B}, \ldots]$ applied to normal equations

and System of Euler equations $\longleftrightarrow$ condensed system

Convergence and complexity analysis for control problem with elliptic PDEs [Dahmen, Kunoth, SICON '05]

'Benchmark' Theorem [Gunzburger, Kunoth, SICON '11]

For any target accuracy $\varepsilon > 0$ SOLVE $[\varepsilon, \mathbf{Q}, \mathbf{g}] \to \mathbf{u}_\varepsilon$ converges in finitely many steps

$\|\mathbf{u} - \mathbf{u}_\varepsilon\| \leq \varepsilon \quad \|\mathbf{y} - \mathbf{y}_\varepsilon\| \lesssim \varepsilon \quad \|\mathbf{p} - \mathbf{p}_\varepsilon\| \lesssim \varepsilon \quad \mathbf{u}_\varepsilon, \mathbf{y}_\varepsilon, \mathbf{p}_\varepsilon$ finitely supported

$\mathbf{u}, \mathbf{y}, \mathbf{p} \in \mathcal{A}^s \Longrightarrow$

$$(\#\,\mathrm{supp}\,\mathbf{u}_\varepsilon) + (\#\,\mathrm{supp}\,\mathbf{y}_\varepsilon) + (\#\,\mathrm{supp}\,\mathbf{p}_\varepsilon) \lesssim \varepsilon^{-1/s}\left(\|\mathbf{u}\|_{\mathcal{A}^s}^{1/s} + \|\mathbf{y}\|_{\mathcal{A}^s}^{1/s} + \|\mathbf{p}\|_{\mathcal{A}^s}^{1/s}\right)$$

$$\|\mathbf{u}_\varepsilon\|_{\mathcal{A}^s} + \|\mathbf{y}_\varepsilon\|_{\mathcal{A}^s} + \|\mathbf{p}_\varepsilon\|_{\mathcal{A}^s} \lesssim \|\mathbf{u}\|_{\mathcal{A}^s} + \|\mathbf{y}\|_{\mathcal{A}^s} + \|\mathbf{p}\|_{\mathcal{A}^s}$$

$\#$flops $\sim \varepsilon^{-1/s}$

Numerical Example for Distributed Elliptic Control Problem (2D)

min $J(y,u)$ $\quad J(y,u) = \frac{1}{2}\|y - y_*\|^2_{H^{1/2}(\Omega)} + \frac{1}{2}\|u\|^2_{L_2(\Omega)}$ $\quad y_* = h_2 \otimes h_2$

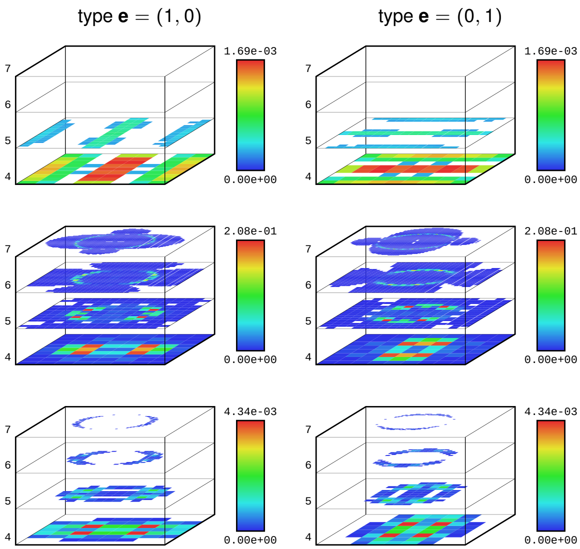
under constraints $\begin{cases} -\Delta y + y = D_{3/2}(h_1 \otimes h_1) + u & \text{in } \Omega := (0,1)^2 \\ \frac{dy}{dn} = 0 & \text{on } \partial\Omega \end{cases}$

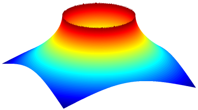
Optimal rate in energy norm ($r = 2$, space dimension $d = 2$, isotropic wavelets) is $\frac{r-1}{d} = \frac{1}{2}$

j	$\|\mathbf{r}_j\|$	#O	#E	#A	#R	S	N_{ad}	$\epsilon_P(y)$	$\epsilon_P(u)$
3								1.31e-02	2.19e-04
4	3.09e-04	1	6	1	11	54.0%	156	1.02e-02	2.19e-04
5	3.55e-04	1	6	2	11	49.0%	534	5.08e-03	2.19e-04
6	1.80e-04	4	4	1	20	51.6%	2182	2.55e-03	2.19e-04
7	1.22e-04	6	6	1	21	43.1%	7169	1.31e-03	2.19e-04
8	5.61e-05	8	8	1	23	36.0%	23745	6.73e-04	2.19e-04
9	2.22e-05	10	9	1	23	30.6%	80525	3.33e-04	1.55e-04
10	1.15e-05	12	9	2	24	27.6%	289790	1.25e-04	1.07e-04
						num. rate $\approx$ 0.55			

[Burstedde, Kunoth '08]

Numerical Example for Elliptic Control Problem (2D)





target state y_*

Numerical Example for One Parabolic PDE

[Chegini, Stevenson '11], [Kunoth, Stapel '13]

Compute $y = y(t,x)$ such that

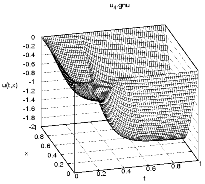
$$
\begin{aligned}
y_t(t,x) - y_{xx}(t,x) &= g(t) \otimes (-\pi^2)\sin(\pi x) && \text{in } I \times \Omega := (0,1)^2 \\
y(t,0) &= y(t,1) = 0 && \text{for } t \geq 0 \\
y(0,x) &= 0 && \text{for } x \in (0,1)
\end{aligned}
$$

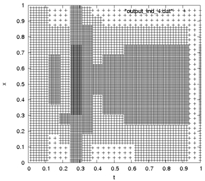
and $g(t) := \begin{cases} 1 & t \in [0, \frac{1}{3}) \\ 2 & t \in [\frac{1}{3}, 1] \end{cases}$

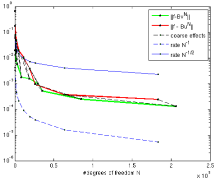
Problem formulation and implementation:

- Modified problem with zero initial conditions $\rightsquigarrow$
  solution space $X = (L_2(I) \otimes H^1(\Omega)) \cap (H^1_{(0}(I) \otimes H^{-1}(\Omega))$ and test space $Y = L_2(I) \otimes V$
- Inhomogeneous initial data: homogenization of initial conditions $\rightsquigarrow$ modification of r.h.s.
- Implementation based on AWM Toolbox by [Vorloeper '10]
  biorthogonal isotropic wavelets of order $m = 2$, $\tilde{m} = 4$
- Iterative solution by GMRES

Plot of Solution, Refined Grid and Residual Error Reduction







8526 degrees of freedom   Expected rate in H^1 (isotropic wavelets): 1/2   red: after coarsening

Summary: Control Problem Constrained by Linear Parabolic PDE

- Fast adaptive solution in space-time formulation based on wavelet methodology
- Uniformly bounded condition numbers of system matrices
- A–posteriori error estimators for coupled system of operator equations
- Automated adaptive refinement for each of state **y**, control **u** and adjoint state **p**
- First convergence proof and algorithmic efficiency: optimal complexity estimates
  method has optimal work/accuracy rate

Extensions and Outlook: Deterministic Control Problems

- Modelling of objective functional (for elliptic PDE constraints: [Dahmen, Kunoth '05], [Burstedde, K. '05])
- Goal-oriented error estimation for elliptic PDEs — convergence and optimal complexity estimates [Dahmen, Kunoth, Vorloeper '05]
- Control problem with nonlinear elliptic PDEs as constraint [Pabel '13]
- Dirichlet boundary control for elliptic PDEs — saddle point systems [Kunoth '05, Pabel '05, Pabel '07]
- Convergence analysis for linear elliptic control problems with inequality constraints on control — Prima-dual-active-set algorithm [Kunoth, Strack '13]

Wavelets $\longleftrightarrow$ Finite Elements

- Optimal preconditioning: multilevel and multigrid methods (for normal equations);
  fast iterative solvers on (non)uniform grids
- (A posteriori) error estimates for PDE constrained control problems [Liu et al . . . et al . . .]
- Convergence theory of adaptive (finite element/DG) method for control problem
  with linear elliptic or parabolic PDE constraints ?
  One or different meshes for all variables ? Refinement / coarsening of meshes ?
  Convergence for one elliptic PDE: [Dörfler '96], [Morin, Nochetto, Siebert '00]
- Complexity estimates ? Optimal complexity ?
  Convergence rates for one elliptic PDE: [Binev, Dahmen, DeVore '04], [Nochetto, Siebert et al '07]

Reformulation of PDE-Constrained Optimal Control Problem

minimize	$J(y,u)$	$= \frac{1}{2}\|y - y_*\|^2_{L_2(I)\otimes V} + \frac{\omega}{2}\|u\|^2_{L_2(I)\otimes V'}$	
subject to	$B\,y$	$= f + u$ (PDE)	$B : X \to Y'$ satisfies (MP)

Necessary and Sufficient Conditions

$$L(y,u,p) := J(y,u) + \langle p,\, By - (f+u)\rangle$$

$$\delta L = 0 \leadsto \begin{array}{rcl} By &=& f+u \\ B^* p &=& R(y_* - y) \\ \omega R^{-1} u &=& p \end{array}$$

$$\iff \begin{pmatrix} R & 0 & B^* \\ 0 & -\frac{1}{\omega}R^{-1} & -I \\ B & -I & 0 \end{pmatrix} \begin{pmatrix} y \\ u \\ p \end{pmatrix} = \begin{pmatrix} Ry_* \\ 0 \\ f \end{pmatrix} \quad \text{(SPP)}$$

$$\leadsto \text{ saddle point operator } \langle Gq, \tilde q\rangle := \left\langle \begin{pmatrix} R & 0 & B^* \\ 0 & -\frac{1}{\omega}R^{-1} & -I \\ B & -I & 0 \end{pmatrix} q, \tilde q \right\rangle$$

symmetric, continuous, boundedly invertible on $\boldsymbol{\mathcal{X}} := X \times U \times X$

$$\Longrightarrow \text{ unique solution } \begin{pmatrix} y \\ u \\ p \end{pmatrix} =: q \text{ of system of PDEs (SPP)}$$

Control problems

- constrained by elliptic or parabolic PDEs; distributed or Neumann/Dirichlet boundary control
- parabolic, parametric evolution operator $B = B(\sigma)$ with

countably many infinite (independent) parameters $\sigma = (\sigma_j)_{j\in\mathbb{N}} \in [-1,1]^{\mathbb{N}}$

minimize $\quad J(y,u) := \mathrm{E}\left[\frac{1}{2}\|y(\sigma) - y_*\|^2_{L_2(I)\otimes V}\right] + \frac{\omega}{2}\mathrm{E}\left[\|u(\sigma)\|^2_{L_2(I)\otimes V'}\right]$

over state $y(\sigma) = y(\sigma; t, x)$ and control $u(\sigma) = u(\sigma; t, x)$

subject to $\quad \mathrm{E}\left[B(\sigma)y(\sigma)\right] = \mathrm{E}\left[u(\sigma)\right] + f \quad$ in V'

uniform probability measure $\rho(\sigma) = \bigotimes_{j\geq 1} \frac{\sigma_j}{2}$ $\qquad$ $\mathrm{E}[u(\sigma)] := \int_{[-1,1]^{\mathbb{N}}} u(\sigma)\, d\rho(\sigma)$

Example: diffusion problems with coefficient $a(\sigma, x)$ expanded by Karhunen-Loève (separation of deterministic and stochastic variables)

$$a(\sigma, x) = \mathrm{E}[a(\sigma, x)] + \sum_{j=1}^{\infty} \sigma_j\, \psi_j(x) \qquad (\psi_j)_{j\in\mathbb{N}} \text{ orthogonal basis}$$

affine parameter dependence

$\rightsquigarrow$ 2nd order elliptic operator $\quad A(\sigma, t) = A_0(t) + \sum_{j=1}^{\infty} \sigma_j\, A_j(t) \quad$ for any $\sigma \in [-1,1]^N$

Necessary conditions for optimality $\rightsquigarrow$ parametric (SPP) $\quad \mathrm{E}\left[G(\sigma)\begin{pmatrix} y(\sigma) \\ u(\sigma) \\ p(\sigma)\end{pmatrix}\right] = g$

Fundamental difficulty

M draws in Monte-Carlo simulation for a single parameter requires

M solutions of (SPP) $\quad \mathrm{E}\left[G(\sigma)\begin{pmatrix} y(\sigma) \\ u(\sigma) \\ p(\sigma)\end{pmatrix}\right] = g$ and slow error rate $M^{-1/2}$ $\quad \rightsquigarrow$ infinitely many σ ?

Wiener/spectral/generalized polynomial chaos expansions: typical assumption of K finitely many σ ("finite-dimensional noise assumption")

Elliptic PDE with random coefficients:
exponential rate of convergence of error in polynomial degree p (with constant depending on K)

[Babuska, Tempone, Zouraris '04]

Control problems with elliptic PDEs [Gunzburger, Lee, Lee '11], [Hou, Lee, Manouzi '11]

New Paradigm for Infinitely Many Parameters

Assumption A: $\{A(\sigma, t) \in \mathcal{L}(V, V') : \sigma \in [-1,1]^{\mathbb{N}}\}$ is regular $\mathfrak{p}$-(real) analytic operator family uniformly for $t \in I$ for some $0 < \mathfrak{p} \leq 1$:

(i) $A(\sigma, t)$ boundedly invertible with uniform bound C_0 w.r.t. $t \in I$ and $\sigma \in [-1,1]^{\mathbb{N}}$

(ii) there exists sequence $b \in \ell^{\mathfrak{p}}(\mathbb{N})$ for some $0 < \mathfrak{p} \leq 1$ such that

$$\text{for all } \nu \in \mathfrak{F}: \quad \sup_{t \in I} \sup_{\sigma \in [-1,1]^{\mathbb{N}}} \left\| (A(0,t))^{-1} (\partial_\sigma^\nu A(\sigma, t)) \right\|_{\mathcal{L}(V,V)} \leq C_0 b^\nu$$

$\mathbb{N}_0^{\mathbb{N}}$ set of all sequences of nonnegative integers

$\mathfrak{F} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} : |\nu| < \infty\}$ set of "finitely supported" such sequences

Assumption A: $\{A(\sigma, t) \in \mathcal{L}(V, V') : \sigma \in [-1,1]^{\mathbb{N}}\}$ is
regular $\mathfrak{p}$-(real) analytic operator family uniformly for $t \in I$ for some $0 < \mathfrak{p} \leq 1$:

(i) $A(\sigma, t)$ boundedly invertible with uniform bound C_0 w.r.t. $t \in I$ and $\sigma \in [-1,1]^{\mathbb{N}}$

(ii) there exists sequence $b \in \ell^{\mathfrak{p}}(\mathbb{N})$ for some $0 < \mathfrak{p} \leq 1$ such that

$$\text{for all } \nu \in \mathfrak{F}: \quad \sup_{t \in I} \sup_{\sigma \in [-1,1]^{\mathbb{N}}} \left\| (A(0,t))^{-1} (\partial_\sigma^\nu A(\sigma, t)) \right\|_{\mathcal{L}(V,V)} \leq C_0 b^\nu$$

$\mathbb{N}_0^{\mathbb{N}}$ set of all sequences of nonnegative integers

$\mathfrak{F} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} : |\nu| < \infty\}$ set of “finitely supported” such sequences

worst situation: $\mathfrak{p} = 1 \rightsquigarrow$ analytic operator family (without additional regularity)

Special case: affine parameter dependence

$$a(\sigma, x) = \mathrm{E}[a(\sigma, x)] + \sum_{j=1}^{\infty} \sigma_j \psi_j(x) \qquad (\psi_j)_{j \in \mathbb{N}} \text{ orthogonal basis}$$

convergence of series $\longleftrightarrow$ summability properties of $(\|\psi_j\|_{L_\infty(\Omega)})_{j \geq 1}$ [Cohen, DeVore, Schwab ’10]

Uniform ellipticity assumption: there exist constants $0 < a_{\min} \leq a_{\max} < \infty$ such that
$a_{\min} \leq a(\sigma, x) \leq a_{\max}$ for all $(\sigma, x) \in [-1,1]^{\mathbb{N}} \times \Omega$

together with $(\|\psi_j\|_{L_\infty(\Omega)})_{j \geq 1} \in \ell^{\mathfrak{p}}(\mathbb{N})$ for $\mathfrak{p} \in (0,1]$ (sparsity class)

$\Longrightarrow$ Assumption A for linear elliptic PDE with $A(\sigma, t) = A_0(t) + \sum_{j \geq 1} \sigma_j A_j(t)$ for same $\mathfrak{p}$

Specifically: finitely many parameters $\Longrightarrow$ Assumption A for arbitrarily small $\mathfrak{p} \in (0,1]$

Theorem: Analyticity of Solution Triple [Kunoth, Schwab '11,'13]

Assumption A for $\{A(\sigma,t) \in \mathcal{L}(V,V') : \sigma \in [-1,1]^{\mathbb{N}}\}$ with some $0 < \mathfrak{p} \le 1$

$\Longrightarrow$

(i) Parametric saddle point operator $G(\sigma)$ continuous & boundedly invertible for all $\sigma \in [-1,1]^{\mathbb{N}}$

(ii) $G(\sigma)$ is regular $\mathfrak{p}$-analytic operator family for same $\mathfrak{p}$

(iii) parametric family of state–control-costate triple $\sigma \mapsto \begin{pmatrix} y(\sigma) \\ u(\sigma) \\ p(\sigma) \end{pmatrix}$ depends analytically on σ

(iii), infinite/high dimension for parameter space

$\rightsquigarrow$ approximation by orthogonal polynomials exhibits exponential rate

$\sigma \in [-1,1]^{\mathbb{N}} \rightsquigarrow$ univariate Legendre polynomial $\tilde{L}_n$ of degree $n \ge 0$ defined via recursion formula

$$(n+1)\tilde{L}_{n+1}(s) := (2n+1)\, s\, \tilde{L}_n(s) - n\tilde{L}_{n-1}(s), \quad s \in (-1,1) \qquad \tilde{L}_0(s) := 1 \text{ and } L_1(s) := s$$

with normalization $\int_{-1}^{1} |L_n(s)|^2 \,\frac{ds}{2} = 1$, i.e., $L_n := (2n+1)^{1/2}\,\tilde{L}_n$

$\rightsquigarrow \quad \{L_n\}_{n\ge 0}$ is orthonormal basis of $L^2(-1,1)$ w.r.t. uniform probability measure

Tensorized Legendre polynomials for $\nu \in \mathfrak{F}$: $\quad L_\nu(\sigma) := \prod_{j\ge 1} L_{\nu_j}(\sigma_j), \ \sigma \in [-1,1]^{\mathbb{N}}$

(only finitely many factors)

Tensorized Legendre polynomials for $\nu \in \mathfrak{F}$: $L_\nu(\sigma) := \prod_{j\geq 1} L_{\nu_j}(\sigma_j),\ \sigma \in [-1,1]^{\mathbb{N}}$

$\leadsto$ countable collection $\mathfrak{L} := \{L_\nu(\sigma) : \nu \in \mathfrak{F}\}$ is Riesz basis for $L^2([-1,1]^{\mathbb{N}}, \rho; \boldsymbol{\mathcal{X}})$:

$\mathfrak{L}$ is orthonormal family in $L^2([-1,1]^{\mathbb{N}}, \rho; \boldsymbol{\mathcal{X}})$ [Gittelson '11]

$\leadsto$ each $v \in L^2([-1,1]^{\mathbb{N}}, \rho; \boldsymbol{\mathcal{X}})$ admits orthonormal expansion

$$v(\sigma) = \sum_{\nu\in\mathfrak{F}} v_\nu L_\nu(\sigma)\ , \qquad v_\nu := \int_{[-1,1]^{\mathbb{N}}} v(\sigma) L_\nu(\sigma) d\rho(\sigma) \in \boldsymbol{\mathcal{X}}$$

and Parseval's equality $\|v\|^2_{L^2([-1,1]^{\mathbb{N}},\rho;\boldsymbol{\mathcal{X}})} = \sum_{\mu\in\mathfrak{F}} \|v_\mu\|^2_{\boldsymbol{\mathcal{X}}}$ holds

Theorem: Simultaneous Approximation of Solution Triple [Kunoth, Schwab '11, '13]

Assumption A for $\{A(\sigma,t) \in \mathcal{L}(V,V') : \sigma \in [-1,1]^{\mathbb{N}}\}$ with some $0 < \mathfrak{p} \leq 1$ $\Longrightarrow$

(iv) admits concurrent expansion in Legendre orthonormal polynomials

$$\begin{pmatrix} y(\sigma) \\ u(\sigma) \\ p(\sigma) \end{pmatrix} = \sum_{\nu\in\mathfrak{F}} \begin{pmatrix} y_\nu \\ u_\nu \\ p_\nu \end{pmatrix} L_\nu(\sigma)$$

(v) $$\left(\left\| \begin{pmatrix} y_\nu \\ u_\nu \\ p_\nu \end{pmatrix} \right\|_{\boldsymbol{\mathcal{X}}} \right)_{\nu\in\mathfrak{F}} \in \ell^{\mathfrak{p}}(\mathfrak{F}) \text{ for same } \mathfrak{p}$$

Proof of (iv): follows from (SPP)

Proof of (v): Prove bounds for partial derivatives of $q(\sigma) = (y(\sigma), u(\sigma), p(\sigma))^T$:

$$\sup_{\sigma\in[-1,1]^{\mathbb{N}}} \|(\partial_\sigma^\nu q)(\sigma)\|_{\boldsymbol{\mathcal{X}}} \leq \frac{C_0}{\ln 2} \|g\|_{\boldsymbol{\mathcal{X}}'} |\nu|!\, b^\nu$$ for all $\nu \in \mathfrak{F}$ by induction with chain rule

Theorem: Simultaneous Approximation of Solution Triple

[Kunoth, Schwab '11, '13]

Assumption A for $\{A(\sigma,t) \in \mathcal{L}(V,V') : \sigma \in [-1,1]^{\mathbb{N}}\}$ with some $0 < \mathfrak{p} \le 1 \implies$

(vi) there exists index set $\Lambda \subset \mathfrak{F}$ of cardinality $\le M$ such that

M-term truncated Legendre expansion $\begin{pmatrix} y_M(\sigma) \\ u_M(\sigma) \\ p_M(\sigma) \end{pmatrix} := \sum_{\nu \in \Lambda} \begin{pmatrix} y_\nu \\ u_\nu \\ p_\nu \end{pmatrix} L_\nu(\sigma)$ allows for

simultaneous generalized polynomial chaos (gpc) approximation

$$\int_{[-1,1]^{\mathbb{N}}} \left\| \begin{pmatrix} y(\sigma) \\ u(\sigma) \\ p(\sigma) \end{pmatrix} - \begin{pmatrix} y_M(\sigma) \\ u_M(\sigma) \\ p_M(\sigma) \end{pmatrix} \right\|_{\boldsymbol{\mathcal{X}}} d\rho(\sigma) \lesssim M^{-(1/\mathfrak{p}-1/2)}$$

on entire parameter domain $[-1,1]^{\mathbb{N}}$

(vii) $$\left\| \mathrm{E}\left[\begin{pmatrix} y(\sigma) \\ u(\sigma) \\ p(\sigma) \end{pmatrix} \right] - \mathrm{E}\left[\begin{pmatrix} y_M(\sigma) \\ u_M(\sigma) \\ p_M(\sigma) \end{pmatrix} \right] \right\|_{\boldsymbol{\mathcal{X}}}$$

$$\le \int_{[-1,1]^{\mathbb{N}}} \left\| \begin{pmatrix} y(\sigma) \\ u(\sigma) \\ p(\sigma) \end{pmatrix} - \begin{pmatrix} y_M(\sigma) \\ u_M(\sigma) \\ p_M(\sigma) \end{pmatrix} \right\|_{\boldsymbol{\mathcal{X}}} d\rho(\sigma) \lesssim M^{-(1/\mathfrak{p}-1/2)}$$

Proof of (vi): by Parseval's identity,

$$\begin{aligned}
\Big\| q(\sigma) - \sum_{\nu\in\Lambda} q_\nu L_\nu(\sigma) \Big\|^2_{L^2([-1,1]^{\mathbb{N}},\rho;\boldsymbol{\mathcal{X}})}
&= \inf\left\{ \|q(\sigma) - v_\Lambda\|^2_{L^2([-1,1]^{\mathbb{N}},\rho;\boldsymbol{\mathcal{X}})} \; : \; v_\Lambda \in \operatorname{span}\Big\{ \sum_{\nu\in\Lambda} v_\nu L_\nu(\sigma) \Big\} \right\} \\
&= \sum_{\nu\notin\Lambda} \|q_\nu\|^2_{\boldsymbol{\mathcal{X}}} \\
&\le M^{-(1/\mathfrak{p}-1/2)} \, \|(\|q_\nu\|_{\boldsymbol{\mathcal{X}}})_{\nu\in\mathfrak{F}}\|_{\ell^{\mathfrak{p}}(\mathfrak{F})}
\end{aligned}$$

(by (v): summability of norms $\|q_\nu\|_{\boldsymbol{\mathcal{X}}}$ of Legendre coefficients by Stechkin's Lemma)

Proof of (vii): from (vi) by triangle and Cauchy-Schwarz inequality

So far: A-Priori Estimates w.r.t. σ

Numerical realization of index set Λ and approximations for one linear elliptic PDE by greedy algorithms [Gittelson '11], [Eigel, Gittelson, Schwab, Zander '13]

Realization for PDE-constrained parametric control problems [Kunoth, Schwab '13], work in progress

Summary

- Control problem constrained by parabolic PDE with
  stochastic coefficients/infinitely many parameters
- Full weak space-time formulation of evolution PDE
- Stochastic coefficient regular $\mathfrak{p}$-analytic for $0 < \mathfrak{p} \leq 1$

  $\rightsquigarrow$ state/control/costate depend analytically on $\mathfrak{p}$

  $\rightsquigarrow$ a-priori estimates:

  simultaneous generalized polynomial chaos approximation
  of state, control, costate of order $-(\frac{1}{\mathfrak{p}} - \frac{1}{2})$

  $\rightsquigarrow$ approximations of mean fields $\mathrm{E}[y(\sigma)], \mathrm{E}[u(\sigma)], \mathrm{E}[p(\sigma)]$ with same order
- Numerical realization of best-M-term generalized polynomial chaos approximation
  for linear elliptic PDEs by greedy algorithms [Gittelson '11], [Eigel, Gittelson, Schwab, Zander '13]
- Multilevel quasi-Monte-Carlo schemes reaching order $\frac{2}{3} < \mathfrak{p} \leq 1$ [Kuo, Schwab, Sloan '12]
- Coupling with adaptive wavelet scheme for space–time discretization
  [Kunoth, Schwab '13, in progress]
- Convergence & complexity analysis for deterministic control problems based on adaptive wavelets Elliptic control problems [Dahmen, Kunoth '05] Control problems with parabolic PDEs [Gunzburger, Kunoth '11]