A: The Kolmogorov-Fokker-Planck equation as a gradient flow
B: Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows
C: Keller-Segel model, a functional analysis approach

Gradient flows, functional inequalities, improvements

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Outline

1. The Kolmogorov-Fokker-Planck equation as a gradient flow
   [JD, B. Nazaret, G. Savaré]

2. Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows
   [JD]

3. Keller-Segel model, a functional analysis approach
   [J. Campos, JD]

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A – The Kolmogorov-Fokker-Planck equation as a gradient flow

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B – Sobolev and Hardy-Littlewood-Sobolev inequalities:

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Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{L^{2^*}(\mathbb{R}^d)} \leq S_d \|\nabla u\|^2_{L^2(\mathbb{R}^d)} \quad \forall \, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{1}$$

and the Hardy-Littlewood-Sobolev inequality

$$S_d \|v\|^2_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx \quad \forall \, v \in L^{\frac{2d}{d+2}}(\mathbb{R}^d) \tag{2}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$. Can we recover this using a nonlinear flow approach ? Can we improve it ?

Keller-Segel model: another motivation [Carrillo, Carlen and Loss] and [Blanchet, Carlen and Carrillo]

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Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0, \quad x \in \mathbb{R}^d \tag{3}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t, \cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1}v \, dx - \mathsf{S}_d \, \|v\|^2_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1} \, dx + \mathsf{S}_d \left(\int_{\mathbb{R}^d} v^{\frac{2d}{d+2}} \, dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}} \, dx$$

where $v = v(t, \cdot)$ is a solution of (3). With the choice $m = \frac{d-2}{d+2}$, we find that $m + 1 = \frac{2d}{d+2}$

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A first statement

Proposition

[J.D.] *Assume that $d \geq 3$ and $m = \frac{d-2}{d+2}$. If v is a solution of* (3) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, *then*

$$\frac{1}{2} \frac{d}{dt} \left[\int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx - \mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \right]$$

$$= \left(\int_{\mathbb{R}^d} v^{m+1} \, dx \right)^{\frac{2}{d}} \left[\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right] \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$

Notice that $u = v^m$ is an optimal function for (1) if v is optimal for (2)

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Improved Sobolev inequality

By integrating along the flow defined by (3), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (1), but only when $d \geq 5$ for integrability reasons

---

Theorem

[J.D.] *Assume that $d \geq 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \leq \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ such that*

$$\mathsf{S}_d \left\| w^q \right\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \left(-\Delta\right)^{-1} w^q \, dx$$

$$\leq \mathcal{C} \left\| w \right\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\left\| \nabla w \right\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \left\| w \right\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

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Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t, x) = c\,(T - t)^\alpha \,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d - 2)\, F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^{d+2}\,|v(x)|$

Lemma

[M. delPino, M. Saez], [J. L. Vázquez, J. R. Esteban, A. Rodríguez]
For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t \to T_-} (T - t)^{-\frac{1}{1-m}}\,\|v(t, \cdot)/\overline{v}(t, \cdot) - 1\|_* = 0$$

with $\overline{v}(t, x) = \lambda^{(d+2)/2}\,\overline{v}_T(t, (x - x_0)/\lambda)$

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Improved inequality: proof (1/2)

$J(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1} \, dx$ satisfies

$$J' = -(m+1) \|\nabla v^m\|_{L^2(\mathbb{R}^d)}^2 \leq -\frac{m+1}{S_d} J^{1-\frac{2}{d}}$$

If $d \geq 5$, then we also have

$$J'' = 2 \, m \, (m+1) \int_{\mathbb{R}^d} v^{m-1} \, (\Delta v^m)^2 \, dx \geq 0$$

Notice that

$$\frac{J'}{J} \leq -\frac{m+1}{S_d} J^{-\frac{2}{d}} \leq -\kappa \quad \text{with} \quad \kappa \, T = \frac{2 \, d}{d+2} \frac{T}{S_d} \left(\int_{\mathbb{R}^d} v_0^{m+1} \, dx \right)^{-\frac{2}{d}} \leq \frac{d}{2}$$

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Improved inequality: proof (2/2)

By the **Cauchy-Schwarz inequality**, we have

$$\frac{J'^2}{(m+1)^2} = \|\nabla v^m\|_{L^2(\mathbb{R}^d)}^4 = \left(\int_{\mathbb{R}^d} v^{(m-1)/2} \, \Delta v^m \cdot v^{(m+1)/2} \, dx \right)^2$$

$$\leq \int_{\mathbb{R}^d} v^{m-1} \left(\Delta v^m \right)^2 \, dx \int_{\mathbb{R}^d} v^{m+1} \, dx = Cst \, J'' \, J$$

so that $Q(t) := \|\nabla v^m(t,\cdot)\|_{L^2(\mathbb{R}^d)}^2 \left(\int_{\mathbb{R}^d} v^{m+1}(t,x) \, dx \right)^{-(d-2)/d}$ is monotone decreasing, and

$$H' = 2 \, J \, (S_d \, Q - 1) \,, \quad H'' = \frac{J'}{J} \, H' + 2 \, J \, S_d \, Q' \leq \frac{J'}{J} \, H' \leq 0$$

$$H'' \leq -\kappa \, H' \quad \text{with} \quad \kappa = \frac{2 \, d}{d+2} \, \frac{1}{S_d} \left(\int_{\mathbb{R}^d} v_0^{m+1} \, dx \right)^{-2/d}$$

By writing that $-H(0) = H(T) - H(0) \leq H'(0) \, (1 - e^{-\kappa \, T})/\kappa$ and using the estimate $\kappa \, T \leq d/2$, the proof is completed $\square$

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C – Keller-Segel model, a functional analysis approach

1. Introduction to the Keller-Segel model
2. Spectral analysis in the functional framework determined by the relative entropy method
3. Collecting estimates: exponential convergence

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Introduction to the Keller-Segel model

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The parabolic-elliptic Keller and Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \ t > 0 \\[2mm] -\Delta v = u & x \in \mathbb{R}^2 \,, \ t > 0 \\[2mm] u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| \, u(t, y) \, dy$$

and observe that

$$\nabla v(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} \, u(t, y) \, dy$$

Mass conservation: $\dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^2} u(t, x) \, dx = 0$

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Blow-up

$M = \int_{\mathbb{R}^2} n_0 \, dx > 8\pi$ and $\int_{\mathbb{R}^2} |x|^2 \, n_0 \, dx < \infty$: blow-up in finite time
a solution u of

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v)$$

satisfies

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \, u(t,x) \, dx$$

$$= - \underbrace{\int_{\mathbb{R}^2} 2\, x \cdot \nabla u \, dx}_{-4\,M} + \frac{1}{2\pi} \underbrace{\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{2x \cdot (y-x)}{|x-y|^2} \, u(t,x) \, u(t,y) \, dx \, dy}_{\frac{(x-y)\cdot(y-x)}{|x-y|^2} \, u(t,x)\, u(t,y)\, dx\, dy}$$

$$= 4\,M - \frac{M^2}{2\pi} < 0 \quad \text{if} \quad M > 8\pi$$

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Existence and free energy

$M = \int_{\mathbb{R}^2} n_0 \, dx \leq 8\pi$: global existence [W. Jäger, S. Luckhaus], [JD, B. Pertha...], [A. Blanchet, JD, B. Perthame], [A. Blanchet, J.A. Carrillo, N. Masmoudi]

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u \, (\nabla (\log u) - \nabla v)]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u \, dx - \frac{1}{2} \int_{\mathbb{R}^2} u \, v \, dx$$

satisfies

$$\frac{d}{dt} F[u(t, \cdot)] = - \int_{\mathbb{R}^2} u \, |\nabla (\log u) - \nabla v|^2 \, dx$$

Log HLS inequality [E. Carlen, M. Loss]: F is bounded from below if $M < 8\pi$

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The dimension $d = 2$

- In dimension d, the norm $L^{d/2}(\mathbb{R}^d)$ is critical. If $d = 2$, the mass is critical
- Scale invariance: if (u, v) is a solution in $\mathbb{R}^2$ of the parabolic-elliptic Keller and Segel system, then

$$\left(\lambda^2\, u(\lambda^2\, t, \lambda\, x), \; v(\lambda^2\, t, \lambda\, x) \right)$$

  is also a solution
- For $M < 8\pi$, the solution vanishes as $t \to \infty$, but saying that *diffusion dominates* is not correct: to see this, study *intermediate asymptotics*

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The existence setting

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \ t > 0 \\[2mm]
-\Delta v = u & x \in \mathbb{R}^2 \,, \ t > 0 \\[2mm]
u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2
\end{cases}
$$

Initial conditions

$$
n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)\,, \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)\,, \quad M := \int_{\mathbb{R}^2} n_0(x)\,dx < 8\,\pi
$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\,dx$ for any $t \geq 0$, see [W. Jäger, S. Luckhaus], [A. Blanchet, JD, B. Perthame]

$v = -\frac{1}{2\pi} \log |\cdot| * u$

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Time-dependent rescaling

$$u(x, t) = \frac{1}{R^2(t)} \, n\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad v(x, t) = c\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1 + 2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\[2mm] c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\[2mm] n(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

[A. Blanchet, JD, B. Perthame] Convergence in self-similar variables

$$\lim_{t \to \infty} \|n(\cdot, \cdot + t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t \to \infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

means *intermediate asymptotics* in original variables:

$$\left\| u(x, t) - \tfrac{1}{R^2(t)}\, n_\infty\left(\tfrac{x}{R(t)}, \tau(t)\right) \right\|_{L^1(\mathbb{R}^2)} \searrow 0$$

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The stationary solution in self-similar variables

$$n_\infty = M \frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2} \, dx} = -\Delta c_\infty , \qquad c_\infty = -\frac{1}{2\pi} \log |\cdot| * n_\infty$$

- Radial symmetry [Y. Naito]
- Uniqueness [P. Biler, G. Karch, P. Laurençot, T. Nadzieja]
- As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\epsilon)|x|^2/2}$ for any $\epsilon \in (0, 1)$ [A. Blanchet, JD, B. Perthame]
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M:

$$\lim_{M \to 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

[D.D. Joseph, T.S. Lundgren] [JD, R. Stańczy]

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The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla \left[n \left(\log n - x + \nabla c \right) \right]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \, dx + \int_{\mathbb{R}^2} \frac{1}{2} |x|^2 \, n \, dx - \frac{1}{2} \int_{\mathbb{R}^2} n \, c \, dx$$

satisfies

$$\frac{d}{dt} F[n(t, \cdot)] = - \int_{\mathbb{R}^2} n \left| \nabla \left(\log n \right) + x - \nabla c \right|^2 \, dx$$

A last remark on 8π and scalings: $n^\lambda(x) = \lambda^2 \, n(\lambda \, x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2) \, dx + \int_{\mathbb{R}^2} \frac{\lambda^{-2} - 1}{2} |x|^2 \, n \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) \, n(y) \log \frac{1}{\lambda} \, dx \, dy$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2M - \frac{M^2}{4\pi} \right)}_{>0 \text{ if } M < 8\pi} \log \lambda + \frac{\lambda^{-2} - 1}{2} \int_{\mathbb{R}^2} |x|^2 \, n \, dx$$

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Keller-Segel with subcritical mass in self-similar variables

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot \left(n \left(\nabla c - x \right) \right) & x \in \mathbb{R}^2 \,, \ t > 0 \\[2mm] c = -\dfrac{1}{2\pi} \, \log | \cdot | * n & x \in \mathbb{R}^2 \,, \ t > 0 \\[2mm] n(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

$$\lim_{t \to \infty} \| n(\cdot, \cdot + t) - n_\infty \|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t \to \infty} \| \nabla c(\cdot, \cdot + t) - \nabla c_\infty \|_{L^2(\mathbb{R}^2)} = 0$$

$$n_\infty = M \, \frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2} \, dx} = -\Delta c_\infty \,, \qquad c_\infty = -\frac{1}{2\pi} \, \log | \cdot | * n_\infty$$

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A parametrization of the solutions and the linearized operator

[J. Campos, JD]
$$-\Delta c = M \, \frac{e^{-\frac{1}{2}|x|^2 + c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}|x|^2 + c} \, dx}$$

Solve
$$-\phi'' - \frac{1}{r}\phi' = e^{-\frac{1}{2}r^2 + \phi}, \quad r > 0$$

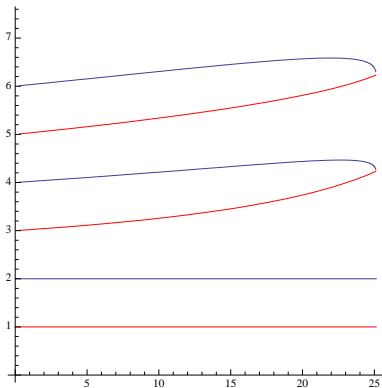with initial conditions $\phi(0) = a$, $\phi'(0) = 0$ and get

$$M(a) := 2\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2}r^2 + \phi_a} \, dx$$

$$n_a(x) = M(a) \, \frac{e^{-\frac{1}{2}r^2 + \phi_a(r)}}{2\pi \int_{\mathbb{R}^2} r \, e^{-\frac{1}{2}r^2 + \phi_a} \, dx} = e^{-\frac{1}{2}r^2 + \phi_a(r)}$$

With $-\Delta \varphi_f = n_a f$, consider the operator defined by

$$\mathcal{L}f := \frac{1}{n_a} \nabla \cdot (n_a (\nabla(f - \varphi_f))) \quad x \in \mathbb{R}^2$$

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Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

Figure: The lowest eigenvalues of $-\mathcal{L}$ (shown as a function of the mass) are 0, 1 and 2, thus establishing that the spectral gap of $-\mathcal{L}$ is 1

[A. Blanchet, JD, M. Escobedo, J. Fernández], [J. Campos, JD], [V. Calvez, J.A. Carrillo], [J. Bedrossian, N. Masmoudi]

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Spectral analysis in the functional framework determined by the relative entropy method

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Simple eigenfunctions

Kernel Let $f_0 = \frac{\partial}{\partial M} c_\infty$ be the solution of

$$-\Delta f_0 = n_\infty f_0$$

and observe that $g_0 = f_0/c_\infty$ is such that

$$\frac{1}{n_\infty} \nabla \cdot \left(n_\infty \nabla (f_0 - c_\infty g_0) \right) =: \mathcal{L} f_0 = 0$$

Lowest non-zero eigenvalues $f_1 := \frac{1}{n_\infty} \frac{\partial n_\infty}{\partial x_1}$ associated with
$g_1 = \frac{1}{c_\infty} \frac{\partial c_\infty}{\partial x_1}$ is an eigenfunction of $\mathcal{L}$, such that $-\mathcal{L} f_1 = f_1$

With $D := x \cdot \nabla$, let $f_2 = 1 + \frac{1}{2} D \log n_\infty = 1 + \frac{1}{2 n_\infty} D n_\infty$. Then

$$-\Delta (D c_\infty) + 2 \Delta c_\infty = D n_\infty = 2 (f_2 - 1) n_\infty$$

and so $g_2 := \frac{1}{c_\infty} (-\Delta)^{-1}(n_\infty f_2)$ is such that $-\mathcal{L} f_2 = 2 f_2$

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Functional setting...

> **Lemma (A. Blanchet, JD, B. Perthame)**
>
> *Sub-critical HLS inequality [A. Blanchet, JD, B. Perthame]*
>
> $$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2} \int_{\mathbb{R}^2} (n - n_\infty)(c - c_\infty) \, dx \geq 0$$
>
> *achieves its minimum for* $n = n_\infty$

$$Q_1[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} F[n_\infty(1 + \varepsilon f)] \geq 0$$

if $\int_{\mathbb{R}^2} f \, n_\infty \, dx = 0$. Notice that f_0 generates the kernel of Q_1

> **Lemma (J. Campos, JD)**
>
> *Poincaré type inequality For any* $f \in H^1(\mathbb{R}^2, n_\infty \, dx)$ *such that*
> $\int_{\mathbb{R}^2} f \, n_\infty \, dx = 0$, *we have*
> $$\int_{\mathbb{R}^2} |\nabla(g \, c_\infty)|^2 \, n_\infty \, dx \leq \int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx$$

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... and eigenvalues

With g such that $-\Delta(g\, c_\infty) = f\, n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1\, f_2\, n_\infty\, dx - \int_{\mathbb{R}^2} f_1\, n_\infty\, (g_2\, c_\infty)\, dx$$

on the orthogonal to f_0 in $L^2(n_\infty\, dx)$ and with $G_2(x) := -\frac{1}{2\pi} \log |x|$

$$Q_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g\, c_\infty)|^2\, n_\infty\, dx \quad \text{with} \quad g = \frac{1}{c_\infty}\, G_2 * (f\, n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L}\, f \rangle = Q_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)$$

Lemma (J. Campos, JD)

$\mathcal{L}$ has pure discrete spectrum and its lowest eigenvalue is 1

A: The Kolmogorov-Fokker-Planck equation as a gradient flow
B: Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows
C: Keller-Segel model, a functional analysis approach

Linearized Keller-Segel theory

$$\mathcal{L} f = \frac{1}{n_\infty} \nabla \cdot \left(n_\infty \nabla (f - c_\infty g) \right)$$

Corollary (J. Campos, JD)

$$\langle f, f \rangle \leq \langle \mathcal{L} f, f \rangle$$

The linearized problem takes the form

$$\frac{\partial f}{\partial t} = \mathcal{L} f$$

where $\mathcal{L}$ is a self-adjoint operator on the orthogonal of f_0 equipped with $\langle \cdot, \cdot \rangle)$. A solution of

$$\frac{d}{dt} \langle f, f \rangle = -2 \langle \mathcal{L} f, f \rangle$$

has therefore exponential decay

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A new Onofri type inequality

[J. Campos, JD]

Theorem (Onofri type inequality)

For any $M \in (0, 8\pi)$, if $n_\infty = M \frac{e^{c_\infty - \frac{1}{2}|x|^2}}{\int_{\mathbb{R}^2} e^{c_\infty - \frac{1}{2}|x|^2} dx}$ with $c_\infty = (-\Delta)^{-1} n_\infty$, $d\mu_M = \frac{1}{M} n_\infty dx$, we have the inequality

$$\log\left(\int_{\mathbb{R}^2} e^\phi \, d\mu_M\right) - \int_{\mathbb{R}^2} \phi \, d\mu_M \le \frac{1}{2M} \int_{\mathbb{R}^2} |\nabla\phi|^2 \, dx \quad \forall\, \phi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2)$$

Corollary (J. Campos, JD)

The following Poincaré inequality holds

$$\int_{\mathbb{R}^2} \left|\psi - \overline{\psi}\right|^2 n_M \, dx \le \int_{\mathbb{R}^2} |\nabla\psi|^2 \, dx \quad \text{where} \quad \overline{\psi} = \int_{\mathbb{R}^2} \psi \, d\mu_M$$

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An improved interpolation inequality (coercivity estimate)

Lemma (J. Campos, JD)

For any $f \in L^2(\mathbb{R}^2, n_\infty \, dx)$ such that $\int_{\mathbb{R}^2} f \, f_0 \, n_\infty \, dx = 0$ holds, we have

$$-\frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) \, n_\infty(x) \log |x - y| \, f(y) \, n_\infty(y) \, dx \, dy$$

$$\leq (1 - \varepsilon) \int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx$$

for some $\varepsilon > 0$, where $g \, c_\infty = G_2 * (f \, n_\infty)$ and, if $\int_{\mathbb{R}^2} f \, n_\infty \, dx = 0$ holds,

$$\int_{\mathbb{R}^2} |\nabla(g \, c_\infty)|^2 \, dx \leq (1 - \varepsilon) \int_{\mathbb{R}^2} |f|^2 \, n_\infty \, dx$$

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Collecting estimates: exponential convergence

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Back to Keller-Segel: exponential convergence for any mass $M \leq 8\pi$

If $n_{0,*}(\sigma)$ stands for the symmetrized function associated to n_0, assume that for any $s \geq 0$

$$\exists\, \varepsilon \in (0, 8\pi - M) \quad \text{such that} \quad \int_0^s n_{0,*}(\sigma)\, d\sigma \leq \int_{B\left(0, \sqrt{s/\pi}\right)} n_{\infty, M+\varepsilon}(x)\, dx$$

Theorem

Under the above assumption, if $n_0 \in L^2_+(n_\infty^{-1}\, dx)$ and $M := \int_{\mathbb{R}^2} n_0\, dx < 8\pi$, then any solution of the rescaled Keller-Segel system with initial datum n_0 is such that

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \frac{dx}{n_\infty(x)} \leq C\, e^{-2t} \quad \forall\, t \geq 0$$

for some positive constant C, where n_∞ is the unique stationary solution with mass M

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Sketch of the proof

- [J. Campos, JD] Uniform convergence of $n(t, \cdot)$ to n_∞ can be established for any $M \in (0, 8\pi)$ by an adaptation of the symmetrization techniques of [J.I. Diaz, T. Nagai, J.M. Rakotoson]

- Uniform estimates (with no rates) easily result

- Estimates based on Duhammel formula inspired by [M. Escobedo, E. Zuazua] allow to prove uniform convergence

- Spectral estimates can be incorporated to the relative entropy approach

- Exponential convergence of the relative entropy follows

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Thank you for your attention !