Relative entropy methods for relaxation limits

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Main motivation for relaxation limits

Hydrodynamic limit for the Boltzmann equation:

$$\nu f_t + \xi \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f) \tag{1}$$

ν: Mach number and ε: Knudsen number

if $\nu = \varepsilon$

$$(1) \longrightarrow \text{Navier–Stokes equations as } \varepsilon \downarrow 0$$

if $\nu = 1$

$$(1) \longrightarrow \text{Euler equations as } \varepsilon \downarrow 0$$

Simplest discrete velocity model (diffusive relaxation): Carleman's equations

$$\begin{cases} \partial_t f_1 + \frac{1}{\varepsilon} \partial_x f_1 = \frac{1}{\varepsilon^2}(f_2^2 - f_1^2) \\ \partial_t f_2 - \frac{1}{\varepsilon} \partial_x f_2 = \frac{1}{\varepsilon^2}(f_1^2 - f_2^2) \end{cases}$$

$\rho = f_1 + f_2$ as $\varepsilon \downarrow 0$ satisfies $\rho_t = \frac{1}{2}(\log(\rho))_{xx}$

Outline

Relative entropy[3]

U (weak, entropy) solution and $\bar{U}$ *smooth solution* of systems of conservation (or balance) laws and (η, q_α) a convex entropy–entropy flux pair

Compute $\eta(U|\bar{U})_t + \sum_\alpha \partial_\alpha q_\alpha(U|\bar{U})$ for

$$\eta(U|\bar{U}) = \eta(U) - \eta(\bar{U}) - \nabla_U \eta(\bar{U}) \cdot (U - \bar{U})$$
$$q_\alpha(U|\bar{U}) = q_\alpha(U) - q_\alpha(\bar{U}) - \nabla_U \eta(\bar{U}) \cdot (F_\alpha(U) - F_\alpha(\bar{U}))$$

This shall lead to a *stability estimate*, used in many different contexts. Recently:

- Hyperbolic relaxation: L., Tzavaras *ARMA* '06; Tzavaras *Commun. Math. Sci.* '05; Berthelin, Vasseur *SIMA* '05; Berthelin, Tzavaras, Vasseur *J. Stat. Physics* '09;
- Weak–strong uniqueness: Demoulini, Stuart, Tzavaras *ARMA*; Feireisl, Novotny *ARMA* '12

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[3] Dafermos, *ARMA '79*; DiPerna, *Indiana U. Math. J.* '79

Outline

Relative entropy for stress relaxation[4]

Hyperbolic–hyperbolic relaxation limit $\varepsilon \downarrow 0$ for

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha S_{i\alpha} \\ \partial_t (S_{i\alpha} - f_{i\alpha}(F)) &= -\frac{1}{\varepsilon}(S_{i\alpha} - T_{i\alpha}(F)) \end{aligned} \tag{2}$$

In (2), i, $\alpha = 1, 2, 3$, F is the deformation gradient, v is the velocity, S is the stress, decomposed as follows:

$$S = f(F) + \int_{-\infty}^{t} \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}(t-\tau)} h(F(\cdot, \tau))\, d\tau$$

Its formal limit is the elasticity system

$$\begin{aligned} \partial_t \bar{F}_{i\alpha} &= \partial_\alpha \bar{v}_i \\ \partial_t \bar{v}_i &= \partial_\alpha T_{i\alpha}(\bar{F}) \end{aligned} \tag{3}$$

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[4] L., Tzavaras *ARMA* '06

Dissipation of mechanical energy

Possible framework:

$$\begin{aligned} T(F) = \nabla_F W(F) = f(F) + h(F)\,, \\ f(F) = \nabla_F W_I(F), \quad h(F) = -\nabla_F W_v(F) \end{aligned} \tag{a}$$

Dissipation of the mechanical energy:

$$\begin{aligned} &\partial_t \left(\frac{1}{2}|v|^2 + \Psi(F, S - f(F)) \right) - \partial_\alpha (v_i S_{i\alpha}) \\ &\quad + \frac{1}{\varepsilon}(F_{i\alpha} - h^{-1}_{i\alpha}(S - f(F)))(S_{i\alpha} - T_{i\alpha}(F)) = 0 \end{aligned} \tag{4}$$

with $\Psi(F, A) = W_I(F) + A \cdot F + G(A)$ and G is a convex function such that $\nabla_A G = -h^{-1}$

$$\begin{aligned} &(F - h^{-1}(S - f(F))) \cdot (S - T(F)) \\ &\quad = (\nabla_A G(A) - \nabla_A G(h(F))) \cdot (A - h(F)) \geq 0 \end{aligned}$$

Relative entropy

Relative entropy $\mathcal{E}_r(v,F,S\,\big|\,\bar v,\bar F,h(\bar F))$ (with the normalization $\Psi(F,h(F))=W(F)$):

$$\mathcal{E}_r := \frac{1}{2}|v-\bar v|^2 + \Psi(F,S-f(F)) - \Psi(\bar F,h(\bar F))$$
$$- \nabla_F\Psi(\bar F,h(\bar F))\cdot(F-\bar F) - \nabla_A\Psi(\bar F,h(\bar F))\cdot(S-f(F)-h(\bar F))$$

Relative entropy identity:

$$\begin{aligned}
&\partial_t\mathcal{E}_r - \partial_\alpha\Big((v_i-\bar v_i)(S_{i\alpha}-T_{i\alpha}(\bar F))\Big)\\
&\quad + \frac{1}{\varepsilon}(F_{i\alpha}-h^{-1}_{i\alpha}(S-f(F)))(S_{i\alpha}-T_{i\alpha}(F))\\
&= (\partial_\alpha\bar v_i)\Big(T_{i\alpha}(F)-T_{i\alpha}(\bar F)-\frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}(\bar F)(F_{j\beta}-\bar F_{j\beta})\Big)\\
&\quad + (\partial_\alpha\bar v_i)(S_{i\alpha}-T_{i\alpha}(F))
\end{aligned} \tag{5}$$

Stability estimate/1

Framework: There exists constants $\gamma_I > \gamma_v > 0$ and $M > 0$ such that

$$\nabla_F^2 W_I(F) \geq \gamma_I I > \gamma_v I \geq \nabla_F^2 (W_I - W)(F) > 0\,, \qquad \text{(b)}$$

$$|\nabla_F^2 W_I(F)| \leq M\,, \quad |\nabla^3 W(F)| \leq M\,, \quad \forall F \qquad \text{(c)}$$

Then

$$\mathcal{E}_r \geq c\big(|v - \bar{v}|^2 + |F - \bar{F}|^2 + |A - h(\bar{F})|^2\big)$$

for a positive $c > 0$. Condition (b) is roughly equivalent to what is called sub-characteristic condition in the theory of relaxation

$$\begin{aligned}
&(F - h^{-1}(S - f(F))) \cdot (S - T(F)) \\
&\quad = (\nabla_A G(A) - \nabla_A G(h(F))) \cdot (A - h(F)) \\
&\quad \geq \frac{1}{\gamma_v}|A - h(F)|^2 = \frac{1}{\gamma_v}|S - T(F)|^2
\end{aligned}$$

Stability estimate/2

Theorem

Let $(v^\varepsilon, F^\varepsilon, S^\varepsilon)$ *smooth solutions of (2) and* $(\bar{v}, \bar{F})$ *smooth solutions of* (3) *defined on* $\mathbb{R}^3 \times [0, T]$*. Then, under* (a), (b), (c), *the relative energy* $\mathcal{E}_r$ *satisfies (5), and, for* $R > 0$, *there exist constants* s *and* $C > 0$ *independent of* ε *such that*

$$\int_{|x|<R} \mathcal{E}_r(x,t)dx \leq C \left(\int_{|x|<R+st} \mathcal{E}_r(x,0)dx + \varepsilon \right)$$

If the data satisfy $\int_{|x|<R+sT} \mathcal{E}_r(x,0)dx \longrightarrow 0$ *as* $\varepsilon \downarrow 0$, *then*

$$\sup_{t\in[0,T]} \int_{|x|<R} \left(|v^\varepsilon - \bar{v}|^2 + |F^\varepsilon - \bar{F}|^2 + |A^\varepsilon - h(\bar{F})|^2 \right) dx \longrightarrow 0$$

as $\varepsilon \downarrow 0$

Outline

Relative entropy and *diffusive* relaxation

When dealing with a *diffusive relaxation*, we must compare (weak, entropy) solutions U of hyperbolic systems of balance laws (with a damping diffusive relaxation term) and *smooth solutions* $\bar{U}$ of its parabolic limit This will lead to two main differences w.r.t. the standard theory:

- the limit $\bar{U}$ is an ε-dependent solution that adapts itself in the relaxation process $\varepsilon \downarrow 0$;
- both solutions U and $\bar{U}$ are energy dissipative, while in the usual theory and energy dissipative (weak) solution is compared to an energy conservative (smooth) solution.

Outline

The model

Isentropic gas dynamics in three space dimensions with a damping term:

$$\begin{cases} \rho_t + \dfrac{1}{\varepsilon} \operatorname{div}_x m = 0 \\ \\ m_t + \dfrac{1}{\varepsilon} \operatorname{div}_x \dfrac{m \otimes m}{\rho} + \dfrac{1}{\varepsilon} \nabla_x p(\rho) = -\dfrac{1}{\varepsilon^2} m, \end{cases} \tag{6}$$

$t \in \mathbb{R}$, $x \in \mathbb{R}^3$, density $\rho \geq 0$ and momentum flux $m \in \mathbb{R}^3$. The pressure $p(\rho)$ satisfies $p'(\rho) > 0$ which makes the system hyperbolic. γ–law: $p(\rho) = k\rho^\gamma$ with $\gamma \geq 1$ and $k > 0$. In the diffusive relaxation limit $\varepsilon \downarrow 0$, solutions of (6) formally converge to those of the porous medium equation

$$\bar{\rho}_t - \triangle_x p(\bar{\rho}) = 0$$

Hilbert's expansion/1

We now use the standard Hilbert's expansion

$$\rho = \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \dots ,$$
$$m = m_0 + \varepsilon m_1 + \varepsilon^2 m_2 + \dots ,$$

into (6) and into

$$\eta(\rho, m)_t + \frac{1}{\varepsilon} \operatorname{div}_x q(\rho, m) = -\frac{1}{\varepsilon^2} \nabla_m \eta(\rho, m) \cdot m = -\frac{1}{\varepsilon^2} \frac{|m|^2}{\rho} \leq 0$$

for the mechanical energy $\eta(\rho, m) = \dfrac{1}{2}\dfrac{|m|^2}{\rho} + h(\rho)$

and the associated flux $q(\rho, m) = \dfrac{1}{2} m \dfrac{|m|^2}{\rho^2} + mh'(\rho)$

$h''(\rho) = \frac{p'(\rho)}{\rho}$; $\rho h'(\rho) = p(\rho) + h(\rho)$

Hilbert's expansion/2

From the equations we get:

$$
\begin{aligned}
&O(\varepsilon^{-1})\ \ \mathrm{div}_x\, m_0 = 0\\
&O(1)\ \partial_t \rho_0 + \mathrm{div}_x\, m_1 = 0\\
&O(\varepsilon)\ \partial_t \rho_1 + \mathrm{div}_x\, m_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
&O(\varepsilon^{-2})\ m_0 = 0\\
&O(\varepsilon^{-1})\ \ -m_1 = \nabla_x p(\rho_0)\\
&O(1)\ \ -m_2 = \nabla_x (p'(\rho_0)\rho_1)\\
&O(\varepsilon)\ \ -m_3 = \partial_t m_1 + \mathrm{div}_x \left(\frac{m_1 \otimes m_1}{\rho_0} \right) + \nabla_x \left(p'(\rho_0)\rho_2 + \frac{1}{2} p''(\rho_0)\rho_1^2 \right)
\end{aligned}
$$

In particular, we recover the equilibrium relation $m_0 = 0$ for the state variables, the Darcy's law $m_1 = -\nabla_x p(\rho_0)$, and that ρ_0 satisfies porous medium equation

Hilbert's expansion/3

From the entropy we get:

$$
\begin{aligned}
&O(1)\ h(\rho_0)_t + \mathrm{div}_x\left(m_1 h'(\rho_0)\right) = -\frac{|m_1|^2}{\rho_0}\\
&O(\varepsilon)\ \partial_t\left(h'(\rho_0)\rho_1\right) + \mathrm{div}_x\left(m_2 h'(\rho_0) + m_1 h''(\rho_0)\rho_1\right)\\
&\qquad = |m_1|^2\frac{\rho_1}{\rho_0^2} - 2\frac{m_1\cdot m_2}{\rho_0}
\end{aligned}
$$

Thus we recover the entropy dissipation relation associated to the porous medium equation for ρ_0

$$
h(\rho)_t - \mathrm{div}_x\left(h'(\rho)\nabla_x p(\rho)\right) = -\frac{|\nabla_x p(\rho)|^2}{\rho}
$$

Outline

Reformulation of the limiting equation

We rewrite $\bar\rho_t - \triangle_x p(\bar\rho) = 0$ as follows

$$\begin{aligned} \bar\rho_t + \frac{1}{\varepsilon}\partial_{x_i}\bar m_i &= 0 \\ \bar m_t + \frac{1}{\varepsilon}\partial_{x_i} f_i(\bar\rho, \bar m) &= -\frac{1}{\varepsilon^2}\bar m + e(\bar\rho, \bar m) \end{aligned} \tag{7}$$

for $(\bar\rho, \bar m = -\varepsilon \nabla_x p(\bar\rho))$ and the *error term*

$$\begin{aligned} \bar e := e(\bar\rho, \bar m) &= \frac{1}{\varepsilon}\,\mathrm{div}_x\left(\frac{\bar m \otimes \bar m}{\bar\rho}\right) - \varepsilon \partial_t \nabla_x p(\bar\rho) \\ &= \varepsilon\,\mathrm{div}_x\left(\frac{\nabla_x p(\bar\rho) \otimes \nabla_x p(\bar\rho)}{\bar\rho}\right) - \varepsilon \nabla_x\left(p'(\bar\rho)\triangle_x p(\bar\rho)\right) \\ &= O(\varepsilon) \end{aligned}$$

Relative entropy/1

The relative entropy is of the form

$$\begin{aligned}\eta(\rho, m\,|\,\bar\rho, \bar m) &:= \eta(\rho, m) - \eta(\bar\rho, \bar m) - \eta_\rho(\bar\rho, \bar m)(\rho - \bar\rho)\\ &\qquad - \nabla_m \eta(\bar\rho, \bar m)\cdot(m - \bar m)\\ &= \frac{1}{2}\rho\left|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right|^2 + h(\rho\,|\,\bar\rho)\end{aligned}$$

while the corresponding relative entropy-flux reads

$$\begin{aligned}q_i(\rho, m\,|\,\bar\rho, \bar m) &:= q_i(\rho, m) - q_i(\bar\rho, \bar m) - \eta_\rho(\bar\rho, \bar m)(m_i - \bar m_i)\\ &\qquad - \nabla_m \eta(\bar\rho, \bar m)\cdot(f_i(\rho, m) - f_i(\bar\rho, \bar m))\\ &= \frac{1}{2} m_i\left|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right|^2 + \rho(h'(\rho) - h'(\bar\rho))\left(\frac{m_i}{\rho} - \frac{\bar m_i}{\bar\rho}\right) + \frac{\bar m_i}{\bar\rho} h(\rho\,|\,\bar\rho)\end{aligned}$$

Relative entropy/2

Proposition

Let (ρ, m) *be a weak entropy solution of* (6) *and let* $(\bar\rho, \bar m)$ *be a smooth solution of* (7). *Then,*

$$\eta(\rho, m\,|\,\bar\rho, \bar m)_t + \frac{1}{\varepsilon}\operatorname{div}_x q(\rho, m\,|\,\bar\rho, \bar m) \leq -\frac{1}{\varepsilon^2} R(\rho, m\,|\,\bar\rho, \bar m) - Q - E,$$

where

$$Q = \frac{1}{\varepsilon}\nabla^2_{(\rho,m)}\eta(\bar\rho, \bar m)\begin{pmatrix}\bar\rho_{x_i}\\ \bar m_{x_i}\end{pmatrix}\cdot\begin{pmatrix}0\\ f_i(\rho, m|\bar\rho, \bar m)\end{pmatrix}$$
$$= -\sum_{i,j}\partial_{x_i x_j} h'(\bar\rho) f_{ij}(\rho, m|\bar\rho, \bar m)$$

$$R(\rho, m\,|\,\bar\rho, \bar m) = \rho\left|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right|^2 \qquad E = e(\bar\rho, \bar m)\cdot\frac{\rho}{\bar\rho}\left(\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\right)$$

Outline

Control of the quadratic term Q

Lemma

If $p(\rho)$ *satisfies* $p''(\rho) \le A\frac{p'(\rho)}{\rho} \ \forall \ \rho > 0$ *for some* $A > 0$, *then* $h(\rho)$ *verifies*

$$p(\rho\,|\bar{\rho}) \le ch(\rho\,|\bar{\rho}) \qquad \forall \ \rho\,,\ \bar{\rho} > 0$$

for a given constant $c > 0$. *Moreover, there exists a* $C > 0$ *such that for any fixed* i

$$|f_i(\rho, m\,|\bar{\rho}, \bar{m})| \le C\eta(\rho, m\,|\bar{\rho}, \bar{m})$$

$$f_{ij}(\rho, m\,|\bar{\rho}, \bar{m}) = \rho\left(\frac{m_i}{\rho} - \frac{\bar{m}_i}{\bar{\rho}}\right)\left(\frac{m_j}{\rho} - \frac{\bar{m}_j}{\bar{\rho}}\right) + p(\rho\,|\bar{\rho})\delta_{ij}$$

Remark: For a γ–law gases:
$\gamma > 1$, $h(\rho) = \frac{1}{\gamma-1}p(\rho)$; $\gamma = 1$, $p(\rho\,|\bar{\rho}) = 0$

Possible framework

We assume

(H$_1$) $\bar\rho$ is a smooth, positive (C^3) periodic solution of the multidimensional porous media equation

$(\rho \geq 0, m)$ is a (periodic) *dissipative weak solution* of (6):

$$\iint_{[0,+\infty)\times\mathbb{T}^3}\left[\left(\frac{1}{2}\frac{|m|^2}{\rho}+h(\rho)\right)\dot{\theta}(t)-\frac{1}{\varepsilon^2}\frac{|m|^2}{\rho}\theta(t)\right]dxdt$$
$$+\int_{\mathbb{T}^3}\left(\frac{1}{2}\frac{|m|^2}{\rho}+h(\rho)\right)\Big|_{t=0}\theta(0)dx \geq 0$$

$$\sup_{t\in(0,T)}\int_{\mathbb{T}^3}\rho dx \leq K_1\,, \qquad \sup_{t\in(0,T)}\int_{\mathbb{T}^3}\left[\frac{1}{2}\frac{|m|^2}{\rho}+h(\rho)\right]dx \leq K_2$$

the pressure $p(\rho)$ satisfies $p''(\rho) \leq A\frac{p'(\rho)}{\rho}\ \forall\ \rho > 0$; for instance, $p(\rho)=\rho^\gamma$, $\gamma \geq 1$

Stability and convergence/1

We denote by

$$\varphi(t) = \int_{\mathbb{T}^3} \eta(\rho, m \,|\, \bar{\rho}, \bar{m})dx$$

Theorem

Let $T > 0$ be fixed. Under hypothesis $(\mathbf{H_1})$, the following stability estimate holds:

$$\varphi(t) \leq C(\varphi(0) + \varepsilon^4), \quad t \in [0, T]\,,$$

where C is a positive constant depending only on T, K_1, $\bar{\rho}$ and its derivatives.
Moreover, if $\varphi(0) \to 0$ as $\varepsilon \downarrow 0$, then

$$\sup_{t\in[0,T]} \varphi(t) \to 0, \;\; as\; \varepsilon \downarrow 0$$

Stability and convergence/2

Proof.

$$\varphi(t)+\frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{T}^3} R(\rho, m\,|\,\bar\rho, \bar m)\,d\tau dx \le \varphi(0)+\int_0^t\int_{\mathbb{T}^3}(|Q|+|E|)\,d\tau dx$$

$$\int_0^t\int_{\mathbb{T}^3}|Q|\,d\tau dx \le C_0\int_0^t \varphi(\tau)\,d\tau$$

$$\int_0^t\int_{\mathbb{T}^3}|E|\,d\tau dx \le \frac{\varepsilon^2}{2}\int_0^t\int_{\mathbb{T}^3}\left|\frac{e(\bar\rho,\bar m)}{\bar\rho}\right|^2\rho\,d\tau dx + \frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^3}\rho\left|\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right|^2 d\tau dx$$

$$\le \varepsilon^4 t C_1 K_1 + \frac{1}{2\varepsilon^2}\int_0^t\int_{\mathbb{T}^3} R(\rho, m\,|\,\bar\rho, \bar m)\,d\tau dx$$

Outline

The model

$$\begin{cases} \rho_t + \frac{1}{\varepsilon} \operatorname{div}_x m = 0 \\ m_t + \frac{1}{\varepsilon} \operatorname{div}_x \frac{m \otimes m}{\rho} + \frac{1}{\varepsilon} \nabla_x p(\rho) = -\frac{1}{\varepsilon^2} m + \frac{1}{\varepsilon} \rho \nabla_x c \\ -\triangle_x c + \beta c = \rho, \end{cases}$$

where $\rho \geq 0$, $c \in \mathbb{R}$, $m \in \mathbb{R}^3$, the pressure $p(\rho)$ satisfies $p'(\rho) \geq 0$ and $\beta > 0$. Easiest case $p(\rho) = \kappa\rho^2$, $\beta > 1/2\kappa$.
Formal limit:

$$\begin{cases} \rho_t + \operatorname{div}_x \left(\rho \nabla_x c - \nabla_x p(\rho) \right) = 0 \\ -\triangle_x c + \beta c = \rho \end{cases}$$

Entropy (in)equalities

Modified entropy–entropy flux pair, based on the mechanical energy of the system:

$$\mathcal{H}(\rho, m, c) = \eta(\rho, m) - \rho c$$
$$\mathcal{Q}(\rho, m, c) = q(\rho, m) - mc.$$

Then the entropy inequality becomes

$$\mathcal{H}(\rho, m, c)_t + \frac{1}{\varepsilon}\operatorname{div}_x \mathcal{Q}(\rho, m, c) \le -\frac{1}{\varepsilon^2}\frac{|m|^2}{\rho} - \rho c_t$$

From the elliptic equation:

$$\rho c_t = \frac{1}{2}\left(\beta c^2 + |\nabla_x c|^2\right)_t - \operatorname{div}_x\left(c_t \nabla_x c\right)$$

Final relation:

$$\left(\mathcal{H}(\rho, m, c) + \frac{1}{2}\left(\beta c^2 + |\nabla_x c|^2\right)\right)_t + \frac{1}{\varepsilon}\operatorname{div}_x\left(\mathcal{Q}(\rho, m, c) - \varepsilon c_t \nabla_x c\right) \le -\frac{1}{\varepsilon^2}\frac{|m|^2}{\rho}$$

Relative entropy estimate

$(\bar\rho, \bar m, \bar c) = \left(\bar\rho, -\varepsilon\bar\rho\nabla_x\big(h'(\bar\rho) - \bar c\big), \bar c\right)$ solves

$$\begin{cases} \bar\rho_t + \dfrac{1}{\varepsilon}\,\mathrm{div}_x\,\bar m = 0 \\ \bar m_t + \dfrac{1}{\varepsilon}\,\mathrm{div}_x\,\dfrac{\bar m\otimes\bar m}{\bar\rho} + \dfrac{1}{\varepsilon}\nabla_x p(\bar\rho) = -\dfrac{1}{\varepsilon^2}\bar m + \dfrac{1}{\varepsilon}\bar\rho\nabla_x\bar c + e(\bar\rho, \bar m) \\ -\triangle_x\bar c + \beta\bar c = \bar\rho \end{cases}$$

Relative entropy estimate:

$$\begin{aligned} &\left(\mathcal{H}(\rho, m, c\,|\,\bar\rho, \bar m, \bar c\,) + \frac{1}{2}\left(\beta(c-\bar c)^2 + |\nabla_x(c-\bar c)|^2\right)\right)_t \\ &+ \frac{1}{\varepsilon}\,\mathrm{div}_x\left(\mathcal{Q}(\rho, m, c\,|\,\bar\rho, \bar m, \bar c\,) - \varepsilon(c-\bar c)_t\nabla_x(c-\bar c)\right) \\ &\leq -\frac{1}{\varepsilon^2}R(\rho, m\,|\,\bar\rho, \bar m\,) - Q - P - E, \end{aligned}$$

where R, Q and E are as before and $P = \frac{1}{\varepsilon}\frac{\bar m}{\bar\rho}(\rho-\bar\rho)\cdot\nabla_x(c-\bar c)$

Outline

The model

$$\begin{cases} u_t - \dfrac{1}{\varepsilon} v_x = 0 \\[2em] v_t - \dfrac{1}{\varepsilon}\tau(u)_x = -\dfrac{1}{\varepsilon^2} v, \end{cases}$$

where τ satisfies the usual conditions $\tau'(u) > 0$ to guarantee strict hyperbolicity. For gas dynamics applications, u denotes the specific volume, v the velocity and

$$\tau\left(\frac{1}{\rho}\right) = -p(\rho),$$

where p stands for the pressure of the gas and ρ for its density.

Formal limit

$$u_t - \tau(u)_{xx} = 0,$$

thus with the relation (Darcy's law) at the limit $v = \tau(u)_x$.

Entropy (in)equalities

$$\mathcal{E}(u,v) = \frac{1}{2}v^2 + \int^u \tau(s)ds = \frac{1}{2}v^2 + W(u),$$

with entropy flux given by

$$\mathcal{F}(u,v) = -v\tau(u)$$

and corresponding entropy inequality

$$\mathcal{E}(u,v)_t + \frac{1}{\varepsilon}\mathcal{F}(u,v)_x \leq -\frac{1}{\varepsilon^2}v^2 \leq 0$$

$\mathcal{E}(u,0) = W(u)$ entropy for the limiting equation:

$$\mathcal{E}(u,0)_t + \mathcal{F}(u,\tau(u)_x)_x = -\big(\tau(u)_x\big)^2$$

Relative entropy estimate

$(\bar{u}, \bar{v}) = (\bar{u}, \varepsilon\tau(\bar{u})_x)$ solves

$$\begin{cases} \bar{u}_t - \dfrac{1}{\varepsilon}\bar{v}_x = 0 \\[2ex] \bar{v}_t - \dfrac{1}{\varepsilon}\tau(\bar{u})_x = -\dfrac{1}{\varepsilon^2}\bar{v} + \varepsilon\tau(\bar{u})_{xt} \end{cases}$$

Relative entropy:

$$\mathcal{E}(u, v \,|\, \bar{u}, \bar{v}) = \frac{1}{2}(v - \bar{v})^2 + W(u \,|\, \bar{u})$$

$$\begin{aligned} \mathcal{E}(u, v \,|\, \bar{u}, \bar{v})_t + \frac{1}{\varepsilon}\mathcal{F}(u, v \,|\, \bar{u}, \bar{v})_x \leq -\frac{1}{\varepsilon^2}(v - \bar{v})^2 + \tau(\bar{u})_{xx}\tau(u \,|\, \bar{u}) \\ - \varepsilon\tau(\bar{u})_{xt}(v - \bar{v}) \end{aligned}$$

Outline

The model

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x - \frac{1}{\varepsilon} z_x = 0 \\ z_t - \frac{\mu}{\varepsilon} v_x = -\frac{1}{\varepsilon^2} z, \end{cases}$$

where $\mu > 0$ and the elastic stress function σ satisfies the usual condition $\sigma'(u) > 0$ which guarantees strict hyperbolicity.

Formal limit:

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = \mu v_{xx}, \end{cases}$$

for which the viscoelastic response is given by $z = \sigma(u) + \mu v_x$

Entropy (in)equalities

$$\mathbb{E}(u,v,z) = \int^{u} \sigma(s)ds + \frac{1}{2}v^2 + \frac{1}{2\mu}z^2 = \Sigma(u) + \frac{1}{2}v^2 + \frac{1}{2\mu}z^2,$$

with entropy flux given by

$$\mathbb{F}_{\varepsilon}(u,v,z) = -(\varepsilon\sigma(u)v + vz)$$

and corresponding entropy inequality

$$\mathbb{E}(u,v,z)_t + \frac{1}{\varepsilon}\mathbb{F}_{\varepsilon}(u,v,z)_x \leq -\frac{1}{\mu\varepsilon^2}z^2 \leq 0$$

$\mathbb{E}(u,v,0) = \Sigma(u) + \frac{1}{2}v^2$ entropy for the limiting system:

$$\mathbb{E}(u,v,0)_t + \mathbb{F}_1(u,v,\sigma(u)_x)_x = -\mu(v_x)^2$$

for

$$\mathbb{F}_1(u,v,\sigma(u)_x) = -(\sigma(u)v + \mu v v_x)$$

Relative entropy estimate

$(\bar{u}, \bar{v}, \bar{z}) = (\bar{u}, \bar{v}, \varepsilon\mu\bar{v}_x)$ solves

$$\begin{cases} \bar{u}_t - \bar{v}_x = 0 \\ \bar{v}_t - \sigma(\bar{u})_x - \frac{1}{\varepsilon}\bar{z}_x = 0 \\ \bar{z}_t - \frac{\mu}{\varepsilon}\bar{v}_x = -\frac{1}{\varepsilon^2}\bar{z} + \varepsilon\mu\bar{v}_{xt} \end{cases}$$

Relative entropy:

$$\mathbb{E}(u, v, z \,|\, \bar{u}, \bar{v}, \bar{z}) = \Sigma(u \,|\, \bar{u}) + \tfrac{1}{2}(v - \bar{v})^2 + \tfrac{1}{2\mu}(z - \bar{z})^2$$

$$\mathbb{E}(u, v, z \,|\, \bar{u}, \bar{v}, \bar{z})_t + \frac{1}{\varepsilon}\mathbb{F}_\varepsilon(u, v, z \,|\, \bar{u}, \bar{v}, \bar{z})_x \leq -\frac{1}{\mu\varepsilon^2}(z - \bar{z})^2 + \bar{v}_x\sigma(u \,|\, \bar{u}) - \varepsilon\bar{v}_{xt}(z - \bar{z})$$

THANK YOU