Nonlinear Geometric Optics

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Google the title to find old .pdf on the web.

Outline

1. Geometric optics pre-Maxwell

2. Geometric optics post-Maxwell

3. Ideas from nonlinear geometric optics

Overview

Geometric optics describes the propagation of light (and other waves).

Using geometric constructions.

Most particularly curves called RAYS.

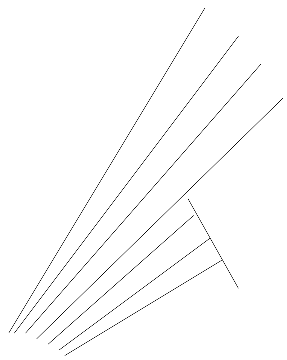
Waves are described as solutions of partial differential equations (PDE). Often hyperbolic.

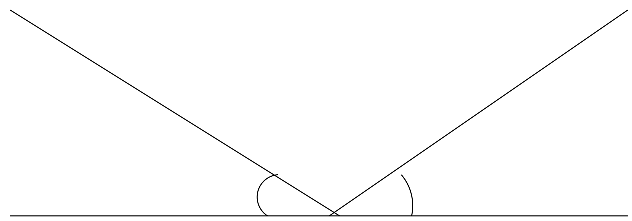
Geometric optics is a family of methods to construct approximate solutions of PDE.

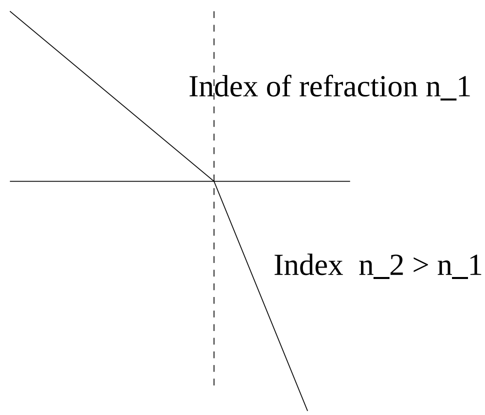
The constructions involve rays, and is simpler than solving the PDE.

This is the wave particle duality. It is not two different descriptions, but two consistent views of a PDE description.

Geometric Optics including Fermat's contributions.







$$\frac{\sin\theta_i}{\sin\theta_r} = \frac{n_2}{n_1}$$

Fermat: Least time, AND, speed $= 1/n$.

Precedes Roemer ($c < \infty$), Newton (d/dx), and, Maupertuis (least acton).

Maxwell/Faraday revolution

Light is a manifestation of electromagnetism.

The fundamental Maxwell equations are,

$$\frac{\partial E}{\partial t} \;=\; \operatorname{curl} B, \qquad \frac{\partial B}{\partial t} \;=\; -\operatorname{curl} E\,.$$

Puzzle: How do the rays relate to these equations?

Initial value problem reasonable, e.g. by marching,

$$E(t+\Delta t) \;\approx\; E(t) \;+\; \Delta t \operatorname{curl} B(t)\,,$$
$$B(t+\Delta t) \;\approx\; B(t) \;-\; \Delta t \operatorname{curl} E(t)\,.$$

(Attn! For this approx. method, $E_{\text{approx}}\big|_{t=1}$ divergent as $\Delta t \to 0$ for nonanalytic initial data.)

Evolution is good thanks to energy conservation,

$$\frac{d}{dt}\int |E(t,x)|^2 + |B(t,x)|^2 \; dx = 0\,.$$

Linear PDE 101; plane wave solutions.

Euler:

$$p(d/dt)\,u \;=\; \left[a_n\Big(\frac{d}{dt}\Big)^n + \cdots a_1\frac{d}{dt} + a_0\right]u \;=\; 0$$

solved by $\quad u = e^{irt} \quad$ for r satisfying $\quad p(ir) = 0\,.$

For scalar constant coefficient PDE,

$$u_{tt} - u_{xx} = 0, \quad u_{tt} + u_{xx} = 0, \quad u_t = u_x, \quad u_t = iu_x$$

insert

$$u = e^{i(t\tau + x\xi)}$$

to find

$$\tau = \pm\xi, \qquad \tau = \pm i\xi, \qquad \tau = \xi, \qquad \tau = i\xi.$$

First and third are stable

Second and fourth are unstable. Have plane waves growing as fast as any exponential. Hadamard's criterion.

Maxwell is a **system** with unknown $u = (E, B) \in \mathbf{R}^6$

$$P(\partial_t, \partial_x)\, u \;=\; \begin{bmatrix} \partial_t & \text{curl} \\ -\text{curl} & \partial_t \end{bmatrix} u \;=\; \left[\partial_t + \sum_1^3 A_j \partial_j\right] u \;=\; 0$$

A_j are symmetric 6×6 real matrices.

$$P(\tau, \xi) \;:=\; \left[\tau I + \sum A_j \xi_j\right]$$

is a degree 1 polynomial with values in 6×6 real matrices.

Plane wave solutions. For $a \in \mathbf{R}^6$,

$$u \;=\; e^{i(t\tau + x.\xi)}\, a\,, \qquad P\,u = i\, e^{i(t\tau + x.\xi)}\, P(\tau, \xi)\, a$$

$$\det P(\tau, \xi) = 0\,, \quad \textbf{eikonal equation} \quad (\tau^2 = |\xi|^2)$$

$$a \in \ker P(\tau, \xi)\,, \quad \textbf{polarization}\,.$$

Compute kernel (dim(ker) = 2) to find

$$\tau = \pm|\xi|\,, \qquad e.\xi = 0\,, \qquad a \,\Big\|\, \left(e\,, \mp \frac{\xi}{|\xi|} \wedge e\right) \;\in\; \mathbb{C}^3 \times \mathbb{C}^3\,.$$

Short wavelength family

Plane wave $e^{i\tau t + x.\xi}\, a$. For t fixed you see $e^{ix.\xi}\, a$

For $|\xi| >> 1$, oscillates with wavelength $O(1/|\xi|) << 1$

Light has short wavelength.

Maxwell is homogeneous, $\epsilon \in \mathbf{R} \setminus 0$ then

$$u(t,x) \ \text{ solves} \quad \Longleftrightarrow \quad u(t/\epsilon\,,\, x/\epsilon) \ \text{ solves}$$

All wavelengths possible. The family

$$u^\epsilon \ := \ e^{i(t\tau + x.\xi)/\epsilon}\, a$$

consists of plane wave solutions with wavelength $O(\epsilon)$.

Discovery method I, slowly varying envelope.

Consider the slowly varying envelope *ansatz*

$$u^\epsilon \ := \ e^{i(t\tau + x.\xi)/\epsilon}\, a(t,x)\,, \qquad \forall \alpha\,, \ \ \partial^\alpha a \text{ is bounded.}$$

An observer on neighborhood of size L, $\epsilon << L << 1$, sees a nearly constant amplitude.

If $P(\tau,\xi)\, a(t,x) = 0$, observer sees a plane wave solution.

So, τ, ξ should satisfy the eikonal equation and a should be accordingly polarised.

Want to glue these infinitesimal ($L << 1$) approximate solutions together to get a good approximate solution.

Discovery method II, exact formulas.

$$\Box u^\epsilon = 0, \quad u_t^\epsilon\big|_{t=0} = 0, \quad u^\epsilon\big|_{t=0} := f(y)\, e^{iy_1/\epsilon}, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Exact solution by Fourier transform in y. $u^\epsilon = u_+^\epsilon + u_-^\epsilon$,

$$u_\pm^\epsilon(t,y) := \int \hat{f}\Big(\eta - \frac{\mathbf{e}_1}{\epsilon}\Big)\; e^{i(y\eta \mp t|\eta|)}\; d\eta\,.$$

New variable $\zeta := \eta - \frac{\mathbf{e}_1}{\epsilon}$, $\quad \eta = (\mathbf{e}_1 + \epsilon\zeta)/\epsilon$

$$u_+^\epsilon(t,y) = \int\; \hat{f}(\zeta)\; e^{iy(\mathbf{e}_1+\epsilon\zeta)/\epsilon}\; e^{-it|\mathbf{e}_1+\epsilon\zeta|/\epsilon}\; d\zeta\,.$$

Taylor approximation

$$|\,\mathbf{e}_1 + \epsilon\zeta\,| = 1 + \epsilon\zeta_1 + \; O(\,\epsilon^2\,|\zeta|^2\,) \quad \approx \quad 1 + \epsilon\zeta_1 \quad \text{yields}$$

$$u^\epsilon(t,y) \;\approx\; e^{i(y_1 - t)/\epsilon}\, b(t,y)\,,$$

$$b(t,y) := \int \hat{f}(\zeta)\, e^{i(y_1\zeta_1 - t\zeta_1)}\, d\zeta\,, \qquad \frac{\partial b}{\partial t} + \frac{\partial b}{\partial y_1} = 0\,.$$

Equation is *simple, geometric,* and*, has no small parameter.*

Rapily oscillating phase. Relatively slowly varying amplitude.

Slowly varying envelope continued.

Consider $P(\partial_t\,,\,\partial_x)\Big[e^{i(t\tau+x.\xi)/\epsilon}\,a(\epsilon,t,x)\Big]$

Terms in ϵ^{-1} vanish when polarization, $P(\tau,\xi)\,a(t,x)=0$, holds. Then,

$$P(\partial_t\,,\,\partial_x)\Big[e^{i(t\tau+x.\xi)/\epsilon}\,a(t,x)\Big] \;=\; e^{i(t\tau+x.\xi)/\epsilon}\,P(\partial_t,\partial_x)a$$

$P(\tau,\xi)\,a(t,x)=0$ **and** $P(\partial_t,\partial_x)\,a=0$ implies $a=\textit{constant}$. NO GOOD.

$$\dim\ker P(\tau,\xi)=2\,.$$

Two unknown functions. $P(\partial)a=0$ is six equations. Extracting two equations is not obvious. There is a systematic algorithm. For constant coefficient systems like Maxwell in vacuum it yields

$$\Big(\partial_t \;-\; \nabla_\xi\tau(\xi).\partial_x\Big)\,a(t,x) \;=\; 0$$

SVEA continued.

$$\Big(\partial_t \ - \ \nabla_\xi \tau(\xi).\partial_x\Big)\, a(t,x) \ = \ 0\,.$$

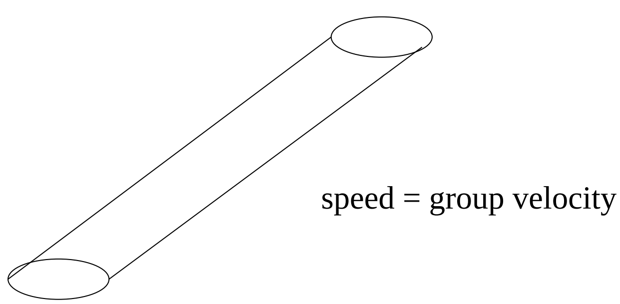
Light ray := integral curve of $\partial_t \ - \ \nabla_\xi \tau(\xi).\partial_x$

Group velocity $\mathbf{v}_g := -\nabla_\xi \tau(\xi)$

For Maxwell, when the amplitude a is constant on rays the local approx solutions glue together to give a good approximate solution.

Take $g(x) \in C_0^\infty$,

$$u_{\rm approx}(t,x) \ := \ e^{i(t\tau + x\xi)/\epsilon}\ g\big(x - \mathbf{v}_g t\big)$$



Approximate solutions are localized and propagate at the group velocity.

Explains the three pre-Maxwell laws of geometric optics.

Thm. *Let* $u^\epsilon_{\rm exact}$ *denote the exact solution of Maxwell's equations which has the same initial data as* $u^\epsilon_{\rm approx}$*, then for any* $T>0$ *and* $\alpha\in\mathbf{N}^{1+3}$

$$\sup_{(t,x)\in[-T,T]\times\mathbf{R}^3} \Big|(\epsilon\partial)^\alpha(u^\epsilon_{\rm exact}-u^\epsilon_{\rm approx})\Big| \;=\; O(\epsilon)\,.$$

Exact solution formula does not work with variable coefficients. Nor for nonlinear problems. Nevertheless one can prove analogous results.

On bounded time intervals. Rmk. diffraction

Lax 1957 strategy. Find correctors

$$u^{\rm new}_{\rm approx} \ \sim\ e^{i(t\tau+x.\xi)/\epsilon}\,\Big(a(t,x)+\epsilon\,a_1(t,x)+\epsilon^2\,a_2(t,x)+\cdots\ \Big)$$

The ϵ^0 term vanishes when $P(\partial)a_0+iP(\tau,\xi)a_1=0$, and this time you can solve.

$$P(\partial)\,u^{\rm new}_{\rm approx}\ =\ O(\epsilon^n)\qquad \text{for all } n$$

$$P(\partial)\big(u_{\rm exact}-u^{\rm new}_{\rm approx}\big)\ =\ O(\epsilon^\infty)$$

Deduce using stability with respect to sources

$$u_{\rm exact}-u^{\rm new}_{\rm approx}=O(\epsilon^\infty)$$

In all the recent results one finds correctors.

Small divisors often obstruct the construction of good correctors.

Related problems

Three other problems with two scales (ϵ and 1) susceptible to similar analyses are *boundary layers*, *internal layers*, and *pulse propagation.*

Close cousins with similar analysis are the *incompressible limit* for fluids and the *quasigeostrophic limit* in geophysics.

Nonlinear optics, the Kerr effect

Recall index of refraction, $n = 1/(\text{speed of light})$

In nonlinear optics speed depends on intensity $|E|^2$,

$$n \;=\; n(|E|) \;=\; n_0 \;+\; n_2|E|^2 \;\;+\;\; \text{h.o.t}$$

Must have $n_2 > 0$ or would violate relativity. High intesity $\mapsto$ slower.

Only with lasers are amplitudes strong enough to see nonlinear effects for light.

Self focussing, rotation of axis of elliptic polarization and harmonic generation at blackboard.

NLGOp constructs approx solutions of NL wave equations

Have form like slowly varying envelope.

Explain the above experiments. And others.

Error estimates as $\epsilon \to 0$.

Two key ideas from NLGeomOpt

1. Amplitude counts, mathematically and physically

$$u^\epsilon_{\rm approx} \;=\; \epsilon^p \Big\{ \quad \text{something } O(1) \quad \Big\}$$

p chosen so the time of nonlinear interaction and time of observation are comparable. Warning: null conditions!

2. Resonance. Suppose that for $1 \le \mu \le K$,

$$\phi_\mu(t,x) = \tau^\mu t + \xi^\mu . x\,, \quad e^{i\phi_\mu(t,x)/\epsilon} \quad \text{are present,}$$

nonlinearity creates

$$e^{i(\sum_{\mu=1}^K n_\mu \phi_\mu)/\epsilon}$$

If $\sum n_\mu \phi_\mu$ satisfies eikonal eq (def: *resonance*), new waves with the new phase will appear. Interacting with each other and the original waves.

Define a projection operator on trigonometric series

$$\mathbf{E} \sum_{n\in\mathbb{Z}^K} a_n(t,x)\; e^{in\phi/\epsilon} \;:=\; \sum_{n\in\mathbb{Z}^K} \pi(n)\, a_n\; e^{in.\phi/\epsilon}\,.$$

where $\pi(n)$ is orthogonal projection on $\ker L(d(n.\phi))$.

E filters nonresonant and nonpolarized parts.

The structure of a typical theorem is

1. Equations for contstructing an approximate solution are derived. They have the form

$$u_{\text{approx}} \;:=\; U(t,x,\phi_1/\epsilon,\ldots,\phi_K/\epsilon)\,,$$
$$U(t,x,\theta) \;\sim\; \sum a_n(t,x)e^{in.\theta}\,,$$

$$\mathbf{E}U \;=\; U\,,$$

and

$$\mathbf{E}L\mathbf{E}U + \mathbf{E}\,F(U) \;=\; 0.$$

Analogue: find two equations for a_0.

2. The equations for U with appropriate initial data are uniquely solvable. Yields a family of candidates for approximation solutions.

3. There is a family of exact solutions so that error of the approximation tends to zero as $\epsilon \to 0$. Proofs use correctors.

The nonlinear term couples distinct resonant combinations. In particular it is possible to start with few combinations and to ignite many.

A striking example of resonance

Thm [JMR]. *For the Euler equations of inviscid compressible fluid flow in dimension $d = 2$, there is a family of smooth exact solutions which at time $t = 0$ has waves with exactly three phases, and, for $t > 0$ has waves oscillating with a countable number of phases whose group velocities have directions dense on the sphere.*

There is conservation of energy, so the incoming energy is spread among the countable number of " outgoing" waves all of the same order in ϵ.

This is an example of complicated nonlinear interaction, not of growth.

The solution is described with error $O(\epsilon^N)$ for all N.

In the applied community of formal asymptotics, such situations were explicitly excluded by finiteness hypotheses. One of the contributions of the rigorous theory was to show that these hypotheses are not needed.