Long-time asymptotics for the Camassa–Holm equation

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The Camassa-Holm equation

$$u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

(another form: $m_t + (m\frac{\partial}{\partial x} + \frac{\partial}{\partial x}m)u = 0$ with $m = u - u_{xx} + \omega$)

- abstract bi-Hamiltonian equation (Fokas, Fuchssteiner, 1981)
- unidirectional propagation of 2-dim waves in shallow water (Camassa, Holm, 1993); constant ω is related to the critical wave speed $\sqrt{gh_0}$
  - g is the gravity acceleration,
  - h_0 is the undisturbed water depth.
- waves in hyperelastic rods (Dai, 1998)
- geodesic flow on diffeomorphism group of the line (Misiolek, 1998)

▷ $\omega = 0$: peakons $c \cdot e^{-|x-ct|}$

▷ $\omega > 0$: smooth solitary waves

Shallow water wave models

- result from approximations to “full” equations (Euler, Green–Naghdi) governing the motion of inviscid fluid whose surface can exhibit gravity wave propagation
- small parameters: $\varepsilon = \frac{a}{h}$, $\mu = \frac{h^2}{\lambda^2}$, where
  - a is the typical amplitude
  - h is the mean depth
  - λ is the typical wavelength

▷ shallow water scaling: $\mu << 1$ weakly dispersive

▷ long-wave: $\varepsilon << 1$ weakly nonlinear

Shallow water wave models

- long-wave regime: $\mu << 1$, $\varepsilon = O(\mu)$; balance between nonlinearity and dispersion
  - ▷ Korteweg-de Vries (KdV) eq.:

$$u_t + u_x + \varepsilon \frac{3}{2} u u_x + \mu \frac{1}{6} u_{xxx} = 0$$

  - ▷ family of Benjamin, Bona, Mahoney (BBM) eqs. (same accuracy as KdV):

$$u_t + u_x + \varepsilon \frac{3}{2} u u_x + \mu(\alpha u_{xxx} + \beta u_{xxt}) = 0$$

- "Camassa–Holm scaling": $\mu << 1$, $\varepsilon = O(\sqrt{\mu})$.
  - more nonlinear than dispersive; could allow breaking waves
  - generalization of the BBM equations (approximation of the same order $O(\mu^2)$ as the BBM)

$$u_t + u_x + \varepsilon \frac{3}{2} u u_x + \mu(\alpha u_{xxx} + \beta u_{xxt}) = \varepsilon\mu(\gamma u u_{xxx} + \delta u_x u_{xx})$$

(some conditions on $\alpha, \beta, \gamma, \delta$)

Integrable models (bi-Hamiltonian structure, Lax pair)

- in the BBM family, only KdV is integrable
- in the "Camassa–Holm scaling": two equations are integrable. After rescaling:
  - ▷ the Camassa–Holm (CH) equation (1993):

$$u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad \omega \in \mathbb{R}$$

  - ▷ the Degasperis–Procesi (DP) equation (1999):

$$u_t - u_{txx} + 3\omega u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad \omega \in \mathbb{R}$$

  - ⋆ in the family of equations (b-family)

$$u_t - u_{txx} + u_x + (b+1)uu_x = bu_x u_{xx} + uu_{xxx},$$

  only integrable are the Camassa–Holm (b=2) and the Degasperis–Procesi (b=3) equations.

Hydrodynamical relevance

- hydrodynamical relevance of CH and DP equations: Johnson (2002), Dullin, Gottwald, Holm (2003), Constantin and Lannes (2009).
- inconsistency of CH with shallow water theory: Blatt, Mikhailov (arXiv, 2010)

CH ($\omega = 1$) on $(-\infty, \infty)$ with fast decaying initial data

Let $u(x,t)$ be the solution of the initial-value problem for the Camassa-Holm equation (for $\omega = 1$):

- $u_t - u_{txx} + 2u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$
- $u(x,0) = u_0(x)$

Let $m(x,t) := u(x,t) - u_{xx}(x,t)$.

Assumptions:

- $u_0(x) \to 0$ as $|x| \to \infty$
- $m(x,0) + 1 > 0$ for all $x \in \mathbb{R}$ (then $m(x,t) + 1 > 0$ for all x, t)

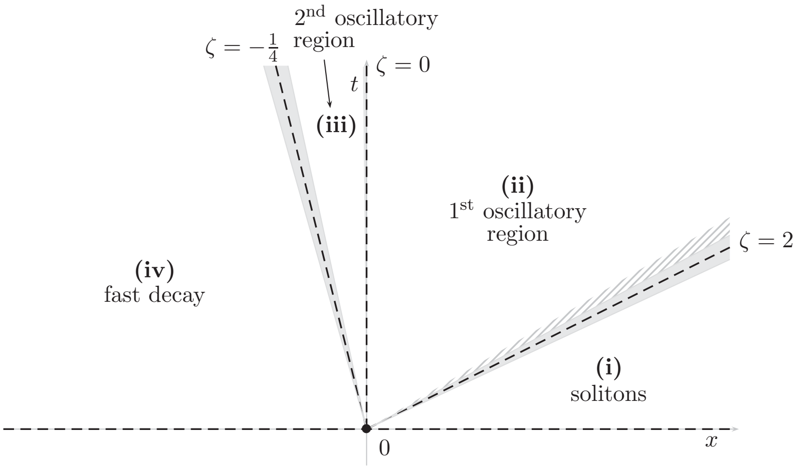
Question

How does $u(x,t)$ behave for large t?

Answer

- four sectors in the (x,t) half-plane where $u(x,t)$ behaves differently for large t, depending on the magnitude of $\zeta = \frac{x}{t}$
- transition zones (2 Painlevé zones; collisionless shock region)

Long-time asymptotics



Four sectors and transition zones in the (x, t)-half-plane, $\zeta = \frac{x}{t}$.
Painlevé zones: (a) $|\zeta - 2| t^{2/3} < C$; (b) $\left|\zeta + \frac{1}{4}\right| t^{2/3} < C$

Long-time asymptotics

Sector (i): $u(x,t)$ looks like a finite train of solitons

Sector (ii): $u(x,t)$ looks like a slowly decaying modulated oscillation

Sector (iii): $u(x,t) \sim$ the sum of two decaying modulated oscillations

Sector (iv): $u(x,t)$ is fast decaying

- transitions: in terms of solutions of Painlevé II equation

$$v''(s) = sv(s) + 2v^3(s)$$

Method

Lax pair:

$$\begin{aligned} \Psi_{xx} &= \frac{1}{4}\Psi + \lambda(m+1)\Psi, \\ \Psi_t &= \left(\frac{1}{2\lambda} - u\right)\Psi_x + \frac{1}{2}u_x\Psi \end{aligned}$$

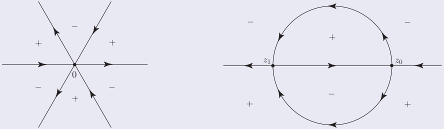
Inverse scattering transform method in the RHP form

- using
  - the Lax pair associated to the CH equation
- construct
  - a multiplicative matrix Riemann-Hilbert problem (RHP)
- obtain
  - a representation of the solution $u(x,t)$ of the CH equation in terms of the solution $\mu(x,t;k)$ of the associated RHP
- obtain
  - the long-time asymptotics of $u(x,t)$ via the Deift-Zhou nonlinear steepest descent method.

Riemann–Hilbert problem

Data

- An oriented contour Σ in the complex plane.



- A jump along Σ, i.e., a function $J\colon \Sigma \to \mathrm{GL}(2,\mathbb{C})$
- Points $\kappa_1,\dots,\kappa_n \notin \Sigma$ with prescribed polar parts.

Problem

Find a matrix-valued function $M(k)$ such that

- M is meromorphic on $\mathbb{C} \setminus \Sigma$. (analyticity condition)
- $M_-(k) = M_+(k)J(k)$ for $k \in \Sigma$. (jump condition)
- M satisfies residue conditions at $\kappa_1,\dots,\kappa_n$. (residue conditions)
- $M(\infty) = I$. (normalization condition)

▷ $J = J(x,t;k) \mapsto M = M(x,t;k)$

▷ analogue of contour integrals: Bessel, Airy, Hermite,. . .

Soliton sector: $x/t > 2$

$u(x,t)$ behaves like the sum of N solitons:

- parameters: $\{\nu_j, \gamma_j\}_{j=1}^N$ (discrete spectrum of x-eqn. from Lax pair)

- $$u(x,t) = \sum_{j=1}^{N} u_j(x - c_j t - x_{0j}) + \mathrm{O}\left(\frac{1}{t^l}\right), \qquad c_j = \frac{2}{1 - 4\nu_j^2}$$

  where $u_j(\cdot)$: 1-soliton ($\nu = \nu_j$) in parametric form (cf. KdV)

  - $u(X) = U(Y(X))$
  - $U(Y) = \dfrac{16\nu^2}{1-4\nu^2} \dfrac{1}{1 + 4\nu^2 + (1-4\nu^2)\cosh Y}$,
  - $X(Y) = Y + \log \dfrac{1 - 2\nu + (1+2\nu)e^{-2\nu Y}}{1 + 2\nu + (1-2\nu)e^{-2\nu Y}}$

  - $$x_{j0} = \log\left[\frac{\gamma_j}{2\nu_j}\frac{1+2\nu_j}{1-2\nu_j}\prod_{l=j+1}^{N}\left(\frac{\nu_j-\nu_l}{\nu_j+\nu_l}\right)^2 \prod_{l=j+1}^{N}\left(\frac{1+2\nu_l}{1-2\nu_l}\right)^2\right]$$

Here $0 < \nu_j < 1/2$ and thus the soliton velocity c_j: $c_j > 2$.

Decaying modulated oscillations ($\zeta = x/t$)

(ii) $0 \leqslant \zeta < 2$:

- $$u(x,t) = \frac{c_1^{(0)}}{\sqrt{t}} \times \sin\left(c_2^{(0)} t + c_3^{(0)} \log t + c_4^{(0)}\right) + \mathrm{o}\left(\frac{1}{\sqrt{t}}\right),$$
    - ▷ $c_l^{(0)} = c_l^{(0)}(\kappa_0(\zeta); \text{scatt.data})$
    - ▷ $\kappa_0^2(\zeta) = \frac{\sqrt{1+4\zeta}-1-\zeta}{4\zeta}$
    - ▷ $u_0(x) \mapsto \{r(k); \{\nu_j, \gamma_j\}\}$

(iii) $-\frac{1}{4} < \zeta < 0$: sum of 2 oscillations

- $$u(x,t) = \sum_{j=0,1} \frac{c_1^{(j)}}{\sqrt{t}} \times \sin\left(c_2^{(j)} t + c_3^{(j)} \log t + c_4^{(j)}\right) + \mathrm{o}\left(\frac{1}{\sqrt{t}}\right),$$
    - ▷ $c_l^{(j)} = c_l^{(j)}(\kappa_0(\zeta), \kappa_1(\zeta); \text{scatt.data})$
    - ▷ $\kappa_1^2(\zeta) = -\frac{\sqrt{1+4\zeta}+1+\zeta}{4\zeta}$

(iv) $\zeta < -1/4$: fast decay

- $u = \mathrm{o}\left(\frac{1}{(|x|+t)^n}\right)$

1^{st} transition zone

close to $x/t = 2$

For $|\frac{x}{t} - 2|t^{2/3} < C$, $C > 0$,

- $u(x,t) \sim -\left(\frac{4}{3}\right)^{2/3} \frac{1}{t^{2/3}} (u^2(s) - u'(s))$,

where

- ▷ $s = 6^{-1/3}(\frac{x}{t} - 2)t^{2/3}$,
- ▷ $u(s)$ is the solution of the Painlevé II equation

$$u''(s) = su(s) + 2u^3(s)$$

fixed by its asymptotics as $s \to +\infty$:

$$u(s) \sim -r(0)\frac{1}{2\sqrt{\pi}} s^{-\frac{1}{4}} \mathrm{e}^{-\frac{2}{3}s^{3/2}}$$

- ▷ $r(k)$ reflection coefficient of x-eqn. from Lax pair

For $\left|\frac{x}{t} + \frac{1}{4}\right| t^{2/3} < C$, $C > 0$:

- $$u(x,t) \sim \frac{12^{1/6}}{t^{1/3}} u_1(s_1) \sin \psi(s_1, t),$$

- ▷ $s_1 = -\left(\frac{16}{3}\right)^{1/3} \left(\frac{x}{t} + \frac{1}{4}\right) t^{2/3}$,
- ▷ $\psi(s_1, t) = -\frac{3\sqrt{3}}{4} t - \frac{3^{5/6}}{2^{4/3}} s_1 t^{1/3} + \Delta$,
- ▷ $u_1(s)$ is the solution of the Painlevé II equation fixed by its asymptotics as $s \to +\infty$:

$$u_1(s) \sim \left| r\left(\frac{\sqrt{3}}{2}\right) \right| \frac{1}{2\sqrt{\pi}} s^{-\frac{1}{4}} \mathrm{e}^{-\frac{2}{3} s^{3/2}}$$

- ▷ Δ: in terms of scattering data

Specific features of RHP for CH

- In order to control analytic properties of eigenfunctions on spectral parameter: two versions of the Lax pair
  - for large k (where $k^2 = -\frac{1}{4} - \lambda$)
  - for k near $\pm\frac{i}{2}$
- Solution $u(x,t)$ of the CH equation: from the evaluation of the solution of the RH problem for k near $\frac{i}{2}$
- Dependence on u of the exponential factor in the RH problem: requires introducing auxiliary scale, which leads to implicit (parametric) formulas for $u(x,t)$ (even for pure soliton solutions)

A 2×2 variant of the Lax pair: $\quad (k^2 = -\frac{1}{4} - \lambda, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$

$$\boxed{\Phi_x + ikp_x\sigma_3\Phi = U\Phi, \qquad \Phi_t + ikp_t\sigma_3\Phi = V\Phi}$$

$U(x,t;k) = U_0(x,t) + \frac{1}{k}U_{-1}(x,t),$
$V(x,t;k) = V_0(x,t) + \frac{1}{k}V_{-1}(x,t) + \frac{k}{4k^2+1}V_1(x,t)$
are 2×2 matrices dependent on $u(x,t)$.

Eigenfunctions: $\hat{\Phi} = \Phi e^{ikp(x,t,k)\sigma_3} \qquad (e^{\hat{\sigma}_3}A \equiv e^{\sigma_3}Ae^{-\sigma_3})$

$$\hat{\Phi}_\pm(x,t,k) = I + \int_{\pm\infty}^{x} e^{-ik(p(x,t,k)-p(y,t,k))\hat{\sigma}_3}(U\hat{\Phi}_\pm)dy$$

$\hat{\Phi} \to I$ as $k \to \infty$.
Here

$$p(x,t,k) := x - \int_x^\infty \left(\sqrt{m(\xi,t)+1} - 1\right) d\xi - \frac{2}{1+4k^2}t$$

$p(x,t,k)$: controls large-k behavior of Φ; depends on u

Eigenfunctions for k near $\pm i/2$ (λ near 0)

Lax pair:

$$\boxed{\Phi^0_x + ik\sigma_3\Phi^0 = U^0\Phi^0, \qquad \Phi^0_t - \frac{4ik}{1+k^2}\sigma_3\Phi^0 = V^0\Phi^0}$$

$$U^0(x,t;k) = -\frac{1+4k^2}{8ik}m(x,t)\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}, \quad U^0(x,t;k)|_{k=\pm i/2} = 0.$$

Eigenfunctions: $\Phi^0 = \hat{\Phi}^0 e^{-ik(x-\frac{2t}{1+4k^2})\sigma_3}$

$$\hat{\Phi}^0_\pm(x,t,k) = I + \int_{\pm\infty}^x e^{-ik(x-y)\hat{\sigma}_3}(U^0\hat{\Phi}^0_\pm)dy$$

$\hat{\Phi}^0_\pm(x,t,\pm\frac{i}{2}) = I$ for all x,t.

In terms of $\hat{\Phi}$: specific matrix structure at $k = \mathrm{i}/2$

$$\begin{pmatrix}\hat{\Phi}^{(1)}_- & \hat{\Phi}^{(2)}_+\end{pmatrix}(x,t,\frac{i}{2}) = F(x,t)\begin{pmatrix} e^{-\frac{1}{2}\int_{-\infty}^x \sqrt{m+1}-1} & 0 \\ 0 & e^{-\frac{1}{2}\int_x^\infty \sqrt{m+1}-1}\end{pmatrix},$$

where $F = \frac{1}{2}\begin{pmatrix} g+1/g & g-1/g \\ g-1/g & g+1/g\end{pmatrix}$, $g = (m+1)^{1/4}$

Scattering functions

- Scattering matrix

$$\hat{\Phi}_+ = \hat{\Phi}_- e^{-ikp(x,t,k)\hat{\sigma}_3} \begin{pmatrix} \bar{a}(k) & b(k) \\ \bar{b}(k) & a(k) \end{pmatrix}, \ k \in \mathbb{R}$$

  - reflection coefficient: $r(k) = \dfrac{b(k)}{\bar{a}(k)}$
- Discrete spectrum: $\{k_j\}_{j=1}^N : a(k_j) = 0$:
  - simple; $N < \infty$
  - $k_j = i\nu_j$, $0 < \nu_j < \frac{1}{2}$
  - $\hat{\Phi}_-^{(1)}(x,t,i\nu_j) = \chi_j e^{2\nu_j p(x,t,i\nu_j)} \hat{\Phi}_+^{(2)}(x,t,i\nu_j)$

$u_0(x)$ determines $a(k), b(k)$

Riemann-Hilbert problem in (x, t)

$$M(x,t,k) := \begin{cases} \begin{pmatrix} \frac{\hat{\Phi}_-^{(1)}}{a(k)} & \hat{\Phi}_+^{(2)} \end{pmatrix}, & \operatorname{Im} k > 0 \\ \begin{pmatrix} \hat{\Phi}_+^{(1)} & \frac{\hat{\Phi}_-^{(2)}}{\bar{a}(\bar{k})} \end{pmatrix}, & \operatorname{Im} k < 0 \end{cases}$$

- $M_- = M_+ e^{-ikp(x,t,k)\hat{\sigma}_3} \begin{pmatrix} 1 & -r(k) \\ \bar{r}(k) & 1-|r(k)|^2 \end{pmatrix}, \quad k \in \mathbb{R}$
- $M \to I$ as $k \to \infty$
- $Res_{k=i\nu} M^{(1)} = i\gamma_j e^{-2\nu_j p(x,t,i\nu_j)} M^{(2)}(x,t,i\nu_j)$
- $M \sim \frac{\alpha(x,t)}{ik} \begin{pmatrix} -c & -1 \\ c & 1 \end{pmatrix}, k \to 0$
- structure at $k = \frac{i}{2}$: $M(x,t,\frac{i}{2}) = \begin{pmatrix} \xi & \eta \\ \eta & \xi \end{pmatrix} \begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix}$

$$p(x,t,k) = x - \int_x^\infty \left(\sqrt{m(\xi,t)+1} - 1 \right) d\xi - \frac{2}{1+4k^2} t$$

Introduce new scale: $y(x,t) := x - \int_x^\infty \left(\sqrt{m(\xi,t)+1} - 1\right) d\xi$.

Then $p = p(y,t) = y - \dfrac{2t}{1+4k^2}$.

RH problem ($y, t; 2 \times 1$): Given $r(k), k \in \mathbb{R}; \{\nu_j, \gamma_j\}_{j=1}^N$ ($0 < \nu_j < \frac{1}{2}, \gamma_j > 0$), find $\mu(y,t,k) = (\mu_1 \ \ \mu_2)$ s.t.

- $\mu_-(y,t,k) = \mu_+(y,t,k) J(y,t,k)$, $k \in \mathbb{R}$,
  where

$$J = \begin{pmatrix} 1 & -r(k) e^{-2ik\left(y - \frac{2t}{1+4k^2}\right)} \\ \bar{r}(k) e^{2ik\left(y - \frac{2t}{1+4k^2}\right)} & 1 - |r|^2 \end{pmatrix}$$

- $\mu \to (1 \ \ 1)$, $k \to \infty$
- $\operatorname{Res}_{k=i\nu} \mu_1(y,t,k) = i\gamma_j e^{-2\nu_j\left(y - \frac{2t}{1-4\nu_j^2}\right)} \mu_2(y,t,i\nu_j)$
- $\mu(-k) = \mu(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Obtaining $u(x,t)$ from solution of RHP

Evaluate $\mu = (\mu_1 \;\; \mu_2)(y,t,k)$ for k near $\dfrac{i}{2}$. Then $u(x,t)$ can be obtained in a parametric form:

- $u(y,t) = \frac{1}{2i} \lim_{k \to \frac{i}{2}} \left(\dfrac{\mu_1(y,t;k)\mu_2(y,t;k)}{\mu_1(y,t;\frac{i}{2})\mu_2(y,t;\frac{i}{2})} - 1 \right) \frac{1}{k - \frac{i}{2}}$

- $x(y,t) = y + \log \dfrac{\mu_1(y,t,i/2)}{\mu_2(y,t,i/2)}$

Nonlinear steepest descent method

Nonlinear steepest descent method (Deift / Zhou) is applicable to RHP whose jump matrix depends on a large parameter t and oscillates with t. Aim: asymptotics of the solution as $t \to \infty$.

Nonlinear steepest descent method for oscillatory RHP

- *Deformations* of contour Σ, *approximation* of jump matrix $J(y,t;k)$ in order to arrive at a new RHP with jump matrix J^* approaching I as $t \to +\infty$ on the most part of the contour (outside stationary phase points)
- *Scaling* near stationary phase points in order to arrive at a model RHP, RH$^{\text{mod}}$, with piecewise constant (in k) jump matrix: $J^* \sim J^{\text{mod}}$ as $t \to +\infty$
- Construct explicit solution of model problem (in terms of special functions)

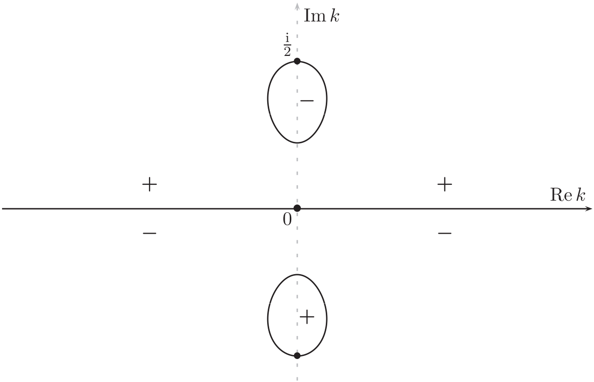
Asymptotics: signature tables (1)

Exponential factor in the jump matrix for RH problem dictate the contour deformations:

$e^{2ik\left(y-\frac{2t}{1+4k^2}\right)} \equiv e^{2it\theta}$, $\theta(y,t,k) = \hat{\zeta}k - \frac{2k}{1+4k^2}$. Here $\hat{\zeta} = \dfrac{y}{t}$.

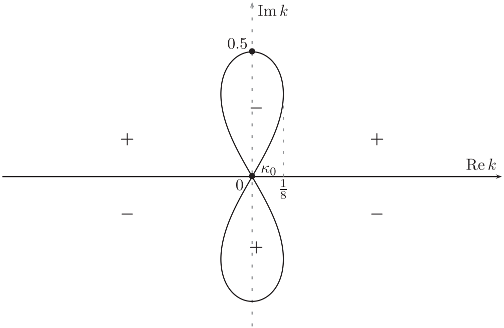
Sign of $\operatorname{Im}\theta$:

(i) $\hat{\zeta} > 2$



Asymptotics: signature tables (2)

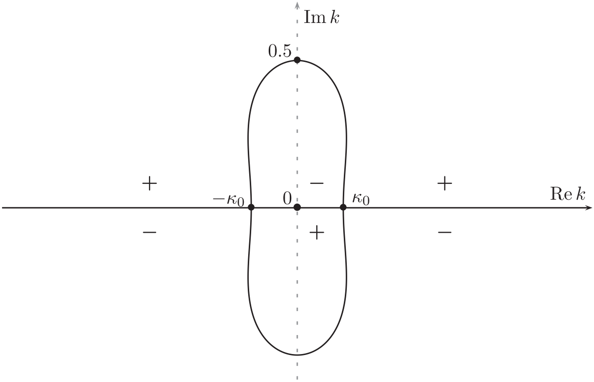
$\hat{\zeta} = 2$



$\kappa_0^2 = \frac{\sqrt{1+4\hat{\zeta}}-1-\hat{\zeta}}{4\hat{\zeta}}$

- 1st Painlevé zone: $\hat{\zeta} \to 2 \quad \Longrightarrow \quad \kappa_0 \to 0$

(ii) oscillatory region: $0 \leqslant \hat{\zeta} < 2$. Here $0 < \kappa_0 < \frac{1}{2}$.

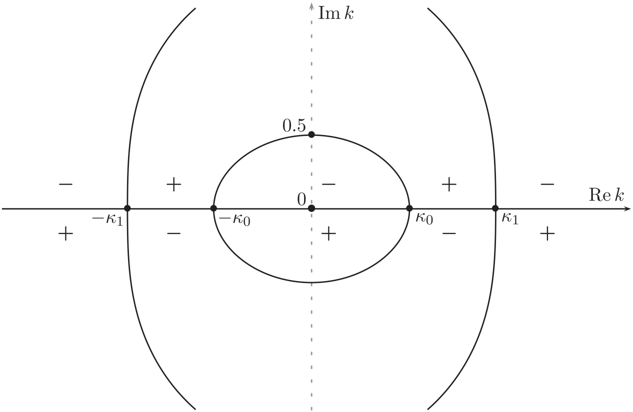


$$\kappa_0^2 = \frac{\sqrt{1+4\hat{\zeta}}-1-\hat{\zeta}}{4\hat{\zeta}}$$

Needs: scalar RHP $\delta_+ = \delta_-(1-|r|^2)$, $-\kappa_0 < k < \kappa_0$

Asymptotics: signature tables (4)

(iii) oscillatory region: $-\frac{1}{4} < \hat{\zeta} < 0$

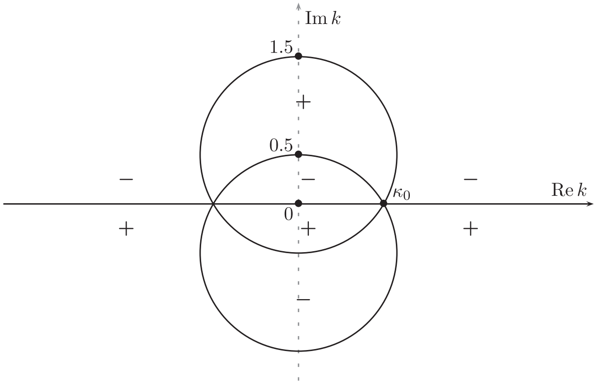


$$\kappa_0^2 = \frac{\sqrt{1+4\hat{\zeta}}-1-\hat{\zeta}}{4\hat{\zeta}}, \ \kappa_1^2 = -\frac{\sqrt{1+4\hat{\zeta}}+1+\hat{\zeta}}{4\hat{\zeta}}$$

Here $\frac{1}{2} < \kappa_0 < \frac{\sqrt{3}}{2} < \kappa_1 < \infty$.

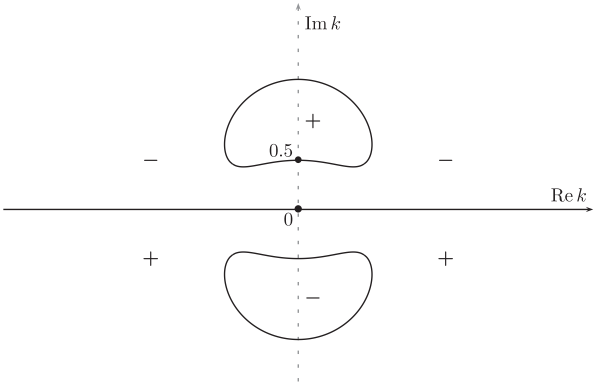
Asymptotics: signature tables (5)

$$\hat{\zeta} = -\frac{1}{4}$$



- 2nd Painlevé zone: $\hat{\zeta} \to -\frac{1}{4} \implies \kappa_j \to \frac{\sqrt{3}}{2}, j = 0, 1$

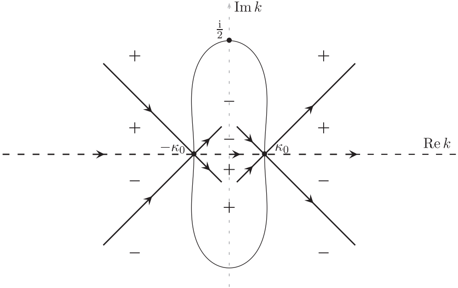
(iv) $\hat{\zeta} < -\dfrac{1}{4}$



Here there are no real stationary points.

RHP deformations

(ii) $0 < x/t < 2$: two crosses



Approximation of phase: $\theta(k) = \theta(\kappa_0) + \alpha(k-\kappa_0)^2\,(1 + O(k-\kappa_0))$

Scaling: $k - \kappa_0 = \frac{\hat{k}}{\sqrt{4\alpha t}}$. Here $\theta(\kappa_0) = -\frac{16\kappa_0^3}{(1+4\kappa_0^2)^2}$, $\quad \alpha = \frac{8\kappa_0(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3}$

Approximation of jump matrix: $\delta(k)e^{-it\theta} \sim \delta_{\kappa_0}\hat{k}^{ih_0}e^{-\frac{i\hat{k}^2}{4}}$, where

- $h_0 = -\frac{1}{2\pi}\log\left(1 - |r(\kappa_0)|^2\right), \qquad \chi(k) = \frac{1}{2\pi i}\int_{-k_0}^{k_0}\log\left(\frac{1-|r(s)|^2}{1-|r(k_0)|^2}\right)\frac{ds}{s-k}$
- $\delta_{\kappa_0} = \left(\frac{128\kappa_0^3(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3}t\right)^{-\frac{ih_0}{2}} e^{i\frac{16\kappa_0^3}{(1+4\kappa_0^2)^2}t} e^{\chi(\kappa_0)}$

Asymptotics in the oscillatory sector $0 < x/t < 2$

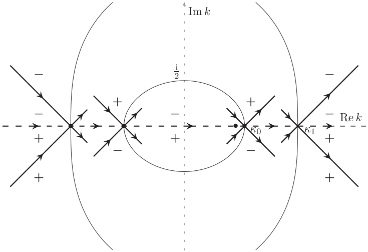
$$\kappa_0 = \kappa_0(\zeta) = \frac{\sqrt{1+4\zeta} - 1 - \zeta}{4\zeta}$$

- $u(x,t) = \frac{c_1^{(0)}}{\sqrt{t}} \cdot \sin\left(c_2^{(0)} t + c_3^{(0)} \log t + c_4^{(0)}\right) + \mathrm{o}\left(\frac{1}{\sqrt{t}}\right),$

  where

  - $c_1^{(0)} = -\left(\frac{32 h_0 \kappa_0}{(3-4\kappa_0^2)(1+4\kappa_0^2)}\right)^{\frac{1}{2}}$
  - $c_2^{(0)} = \frac{32\kappa_0^3}{(1+4\kappa_0^2)^2}$
  - $c_3^{(0)} = -h_0$
  - $c_4^{(0)} = \frac{\pi}{4} - \arg r(\kappa_0) + \arg\Gamma(ih_0) + \frac{1}{\pi}\int_{-\kappa_0}^{\kappa_0} \log|\kappa_0 - s| d\log(1-|r|^2) - h_0 \log\frac{128\kappa_0^3(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3} + \frac{4}{\pi}\kappa_0 \int_{-\kappa_0}^{\kappa_0} \frac{\log(1-|r(\kappa_0)|^2)}{1+4s^2} ds + 4\sum_{j=1}^{N} \arctan\left(\frac{\nu_j}{\kappa_0}\right) + 4\kappa_0 \sum_{j=1}^{N} \log\frac{1+2\nu_j}{1-2\nu_j}$

(iii) $-1/4 < x/t < 0$: four crosses



- $u(x,t) = \sum_{j=0,1} \frac{c_1^{(j)}}{\sqrt{t}} \cdot \sin\left(c_2^{(j)} t + c_3^{(j)} \log t + c_4^{(j)}\right) + \mathrm{o}\left(\frac{1}{\sqrt{t}}\right),$

  where

  - $c_1^{(j)} = -\left(\frac{32 h_j \kappa_j}{|3-4\kappa_j^2|(1+4\kappa_j^2)}\right)^{\frac{1}{2}}$ $\qquad \kappa_0(\zeta) = \frac{\sqrt{1+4\zeta}-1-\zeta}{4\zeta}$
  - $c_2^{(j)} = \frac{32\kappa_j^3}{(1+4\kappa_j^2)^2}$ $\qquad \kappa_1(\zeta) = -\frac{\sqrt{1+4\zeta}+1+\zeta}{4\zeta}$
  - $c_3^{(j)} = (-1)^j h_j$
  - $c_4^{(j)} = c_4^{(j)}(\kappa_0, \kappa_1; r(k); \{\nu_j\}) = \ldots$

Painlevé zone $\left|\frac{x}{t}-2\right| t^{2/3} < C$

Scaling

- $s = 6^{-1/3}\left(\frac{y}{t}-2\right)t^{2/3}, \qquad \hat{k} = (6t)^{1/3}k$

Then $e^{t\theta} \sim e^{\frac{4}{3}\hat{k}^3 + s\hat{k}}$ as in the RHP for the Painlevé II eq.

- $u(x,t) \sim -\left(\frac{4}{3}\right)^{2/3}\frac{1}{t^{2/3}}(v^2(s) - v'(s)),$
  - $v(s)$ solves Painlevé II and

$$v(s) \sim -r(0)Ai(s) \sim -r(0)\frac{1}{2\sqrt{\pi}}s^{-\frac{1}{4}}e^{-\frac{2}{3}s^{3/2}}$$

as $s \to \infty$

Painlevé zone $\left|\frac{x}{t}+\frac{1}{4}\right| t^{2/3} < C$

- $s_1 = -\left(\frac{16}{3}\right)^{1/3}\left(\frac{y}{t}+\frac{1}{4}\right) t^{2/3}, \qquad \hat{k} = -\left(\frac{3t}{16}\right)^{1/3}\left(k \mp \frac{\sqrt{3}}{2}\right)$
  Then $t\theta \sim t\theta\left(\pm\frac{\sqrt{3}}{2}\right) + \frac{4}{3}\hat{k}^3 + s_1\hat{k}$

■ $$u(x,t) \sim \frac{12^{1/6}}{t^{1/3}} v_1(s_1)\sin\psi(s_1,t),$$

- $\psi(s_1,t) = -\frac{3\sqrt{3}}{4}t - \frac{3^{5/6}}{2^{4/3}} s_1 t^{1/3} + \Delta,$
- $v_1(s)$ solves PII and $v_1(s) \sim \left| r\left(\frac{\sqrt{3}}{2}\right)\right| \frac{1}{2\sqrt{\pi}} s^{-\frac{1}{4}} e^{-\frac{2}{3}s^{3/2}},$
- $\Delta = -\frac{4\sqrt{3}}{\pi}\int_0^\infty \frac{\log(1-|r(\xi)|^2)}{1+4\xi^2}d\xi + \arg r\left(\frac{\sqrt{3}}{2}\right) +$
  $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\log(1-|r(\xi)|^2)}{\xi-\frac{\sqrt{3}}{2}}d\xi - 4\sum_{j=1}^{N}\arg\left(\frac{\sqrt{3}}{2}+i\kappa_j\right) - 2\sqrt{3}\sum_{j=1}^{N}\log\left(\frac{1+2\kappa_j}{1-2\kappa_j}\right)$

CH on the half-line for fast decaying IB data

Let $u(x,t)$ be the solution of the initial-boundary-value problem for the Camassa-Holm equation (for $\omega = 1$):

- $u_t - u_{txx} + 2u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$.
- $u(x,0) = u_0(x)$.
- $u(0,t) = v_0(t)$, $u_x(0,t) = v_1(t)$, $u_{xx}(0,t) = v_2(t)$ compatible

▷ $u_0(x) \to 0$ as $x \to +\infty$, $v_j(t) \to 0$ as $t \to \infty$

There are

- two sectors in the (x,t) quarter-plane where $u(x,t)$ behaves differently for large t, depending on the magnitude of $\zeta = \frac{x}{t}$
- transition zones

The structure of the asymptotics is similar to the whole line case, but parameters are determined by initial and boundary conditions.

Half-line problem: $u(0,t) \leqslant 0$

Eigenfunctions:

$$\hat{\Phi} = I + \int_{(x^*,t^*)}^{(x,t)} e^{-ik(p(x,t,k)-p(y,\tau,k))\hat{\sigma}^3} \left(U\hat{\Phi}dy + V\hat{\Phi}d\tau\right),$$

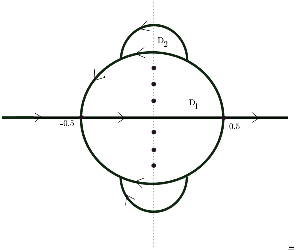
where $(x^*, t^*) \in \{(\infty, 0), (0,0), (0,\infty)\}$,

$$p(x,t,k) = \int_0^x \sqrt{m(\xi,t)+1}d\xi - \int_0^t u(0,\tau)\sqrt{m(0,\tau)+1}d\tau + \frac{t}{2\lambda}$$

Outline:

- $u(x,0) = u_0(x) \mapsto \{a(k), b(k)\}$ (scattering for x-equ. from Lax pair)
- $\begin{pmatrix} u(0,t) = v_0(t) \\ u_x(0,t) = v_1(t) \\ u_{xx}(0,t) = v_2(t) \end{pmatrix} \mapsto \{A(k), B(k)\}$ (scattering for t-equ.)
- $\{a(k), b(k); A(k), B(k)\} \mapsto$ RHP
- compatibility of $\{u_0, v_0, v_1, v_2\}$: $A(k)b(k) - B(k)a(k) = 0, k \in D$

$D = \mathbb{C}_+ \setminus D_1$



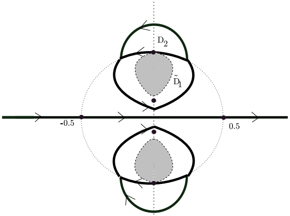
Define $d(k) = a(k)\bar{A}(\bar{k}) - b(k)\bar{B}(\bar{k})$; $\quad R(k) = \frac{\bar{B}(\bar{k})}{a(k)d(k)}, \, k \in D_1$

- Residue conditions: at zeros of $d(k)$: $\hat{\nu}_j, \hat{\gamma}_j$
- Jump matrix:
  - $\begin{pmatrix} 1 & 0 \\ R(k)e^{2ikp} & 1 \end{pmatrix}, \, |k| = \frac{1}{2}, \mathrm{Im}\, k > 0$
  - $\begin{pmatrix} 1 & \bar{r}(k)e^{-2ikp} \\ -r(k)e^{2ikp} & 1-|r|^2 \end{pmatrix}, \, |k| > \frac{1}{2}, \mathrm{Im}\, k = 0$
  - $\begin{pmatrix} 1 & 0 \\ (R-r)e^{2ikp} & 1 \end{pmatrix} \begin{pmatrix} 1 & (\bar{r}-\bar{R})e^{-2ikp} \\ 0 & 1 \end{pmatrix}, \, |k| < \frac{1}{2}, \mathrm{Im}\, k = 0$

Contour deformations

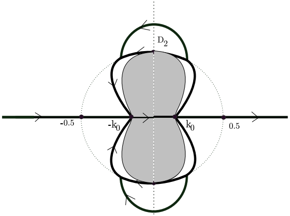
Initial contour deformation: according to signature table.

(i) $x/t > 2$



Soliton parameters: $\{\nu_j, \gamma_j\} \mapsto \{\hat{\nu}_j, \hat{\gamma}_j\}$

(ii) $0 < x/t < 2$



Parameters in asymptotic formulas: $r(k) \mapsto r(k) - R(k)$.

Completely integrable case: $u(0,t)=0$

- symmetry: $A(k) = A\left(\frac{1}{4k}\right)$, $B(k) = B\left(\frac{1}{4k}\right)$
- global relation: $A(k)b(k) - B(k)a(k) = 0$, $\operatorname{Im} k > 0, |k| > \frac{1}{2}$

allow solving $A(k)$, $B(k)$ in terms of $a(k)$, $b(k)$

- $R(k) = \dfrac{\bar{b}(1/4\bar{k})}{a(k)\Delta(k)}$,
  where $\Delta(k) = a(k)\bar{a}\left(\frac{1}{4\bar{k}}\right) - b(k)\bar{b}\left(\frac{1}{4\bar{k}}\right)$
- residue conds.: at zeros $\{\lambda_j\}$ of $\Delta(k)$:

$$\operatorname{Res}_{k=\lambda_j} \mu_1(y,t,k) = \frac{\bar{b}(1/4\bar{\lambda}_j)}{a(\lambda_j)\dot{\Delta}(\lambda_j)} e^{2i\lambda_j p} \mu_2(y,t,\lambda_j)$$