Existence theory for non linear transport equations

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What are non linear transport equations?

Classical, linear transport equation reads

$$\partial_t n(t,x) + \operatorname{div}\left(a(t,x)\, n(t,x)\right) = 0,$$

where the velocity field a is either given or is related to n through another equation.
Recently new models were introduced in various settings (traffic flow for cars or pedestrian, movement of bacteria/cells...) taking local non linear effects into account

$$\partial_t n(t,x) + \operatorname{div}\left(a(t,x)\, f(n(t,x))\right) = 0, \quad t \in \mathbb{R}_+,\ x \in \mathbb{R}^d$$

The function f is given and typically decreases as the density increases.

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As before the field a is given or related to n, for instance

$$u = -\nabla\phi, \quad -\Delta\phi = g(n),$$

again for a non linear function g of n. See Topaz-Bertozzi, Burger, Dolak, Schmeiser and Dalibard-Perthame.

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As before the field a is given or related to n, for instance

$$u = -\nabla\phi, \quad |\phi|^2 - \Delta\phi = g(n),$$

still for a non linear function g of n. See DiFrancesco, Markowich, Pietschmann.

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Many other models of the same type have been derived, see Goatin, Maury, Rascle...

Existence theory

Take for example a vanishing viscosity approximation

$$\partial_t n_\varepsilon(t,x) + \operatorname{div}\left(a_\varepsilon(t,x)\, f(n_\varepsilon(t,x))\right) - \varepsilon^2\, \Delta_x n_\varepsilon = 0,$$
$$n_\varepsilon(t=0,x) = n_\varepsilon^0(x), \quad x \in \mathbb{R}^d.$$

With a_ε given or coupled to n_ε, can we pass to the limit in this equation to obtain a weak solution?

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Easy a priori estimates

- By the maximum principle, in general all L^p bounds are propagated

$$\|n_\varepsilon(t,.)\|_{L^p(\mathbb{R}^d)} \leq C_t\, \|n_\varepsilon^0(.)\|_{L^p(\mathbb{R}^d)}, \qquad n_\varepsilon(t,.) \geq 0 \text{ if } n^0 \geq 0.$$

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- Taking for example $a_\varepsilon = -\nabla\phi_\varepsilon$ with $-\delta\phi_\varepsilon = g(n)$, one has compactness on a_ε

$$\|\operatorname{div} a_\varepsilon\|_{L^\infty} \leq C_g\, \|n_\varepsilon\|_{L^\infty},$$
$$\|a_\varepsilon\|_{W^{1,p}} \leq C_g\, \|n_\varepsilon\|_{L^p} \qquad 1 < p < \infty.$$

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How to obtain compactness on n_ε ?

State of the art

Several existence results are already available in restricted settings.

- Existence (even of strong solutions) for short times is relatively easy. It is only valid up to the first shock.
- The 1-d case can usually be solved with compensated compactness or other regularizing effects.
- For some very precise couplings like

$$\partial_t n(t,x) - \text{div}\,(\nabla\phi(t,x)\, f(n(t,x))) = 0, \quad -\Delta\phi = n,$$

  gradient flows techniques can be used (see Dolbeaut, Maury, Santambroggio).
- For the case

$$\partial_t n(t,x) - \text{div}\,(\nabla\phi(t,x)\, f(n(t,x))) = 0, \quad -\Delta\phi = g(n),$$

  compactness has been proved by Dalibard-Perthame using the rigidity in the kinetic formulation.

General assumptions

Let us summarize the general framework we would like to work in. Consider for n_ε^0 compact and uniformly bounded in $L^1 \cap, L^\infty$ a sequence of solutions to

$$\partial_t n_\varepsilon(t,x) + \operatorname{div}\left(a_\varepsilon(t,x)\, f(n_\varepsilon(t,x))\right) - \varepsilon^2 \Delta_x n_\varepsilon = 0,$$
$$n_\varepsilon(t=0,x) = n_\varepsilon^0(x), \quad x \in \mathbb{R}^d.$$

Assume

$$\exists\, p > 1, \qquad \sup_\varepsilon \sup_{t\in[0,\ T]} \|a_\varepsilon(t,.)\|_{W^{1,p}(\mathbb{R}^d)} < \infty,$$
$$\sup_\varepsilon \|\operatorname{div}_x a_\varepsilon\|_{L^\infty([0,\ T]\times\mathbb{R}^d)} < \infty.$$

And

$$\begin{cases} \operatorname{div}_x a_\varepsilon = d_\varepsilon + r_\varepsilon \quad \text{with } d_\varepsilon \text{ compact and} \\ \exists C > 0,\ s.t.\ \forall \varepsilon > 0, \forall x, y, \\ |r_\varepsilon(x) - r_\varepsilon(y)| \leq C\,|n_\varepsilon(t,x) - n_\varepsilon(t,y)|. \end{cases}$$

The main result

Theorem

Under the previous assumptions, the sequence n_ε is compact in L^1_{loc} and if $a_\varepsilon \longrightarrow a$, it converges to the entropy solution to

$$\partial_t n(t,x) + div\big(a(t,x)\, f(n(t,x))\big) = 0,$$
$$n(t=0,x) = \lim n^0_\varepsilon(x), \quad x \in \mathbb{R}^d.$$

Entropy solution means the usual: For any χ convex and smooth, there exists η s.t.

$$\partial_t \chi(n(t,x)) + \mathrm{div}\,\big(a(t,x)\, \eta(n(t,x))\big) \le 0.$$

A transparent example

The case where $a(t,x) = a(x)$ with $\operatorname{div} a = 0$ is straightforward. Define the flow

$$\partial_t X(t,x) = a(X(t,x)), \quad X(0,x) = x,$$

and introduce

$$g(t,s,x) = n(t, X(s,x)).$$

Then g solves the $1-d$ scalar conservation law

$$\partial_t g + \partial_s (f(g)) = 0.$$

Compactness follows from the usual theory for linear transport equations and scalar conservation laws (with some complication with the Laplacian).

The difficulty in the general case

The problem is that the usual methods to obtain compactness either for linear transport equations or SCL are not compatible.
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- Compensated compactness or other regularizing effects require a non degeneracy assumption.
- The propagation of BV bounds is false for linear transport equations.
- In general using L^1 contraction does not work because $n(t, x+h)$ is not a solution if n is.

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- DiPerna-Lions renormalized solutions require to pass to the limit and use the uniqueness of the limit. Here we do not know how to pass to the limit...
- Methods based on the characteristics (Crippa-DeLellis, Hauray-LeBris-Lions) do not work well with shocks.

A new norm

Define

$$\|n\|_{h,p}^p = \int_{\mathbb{R}^{2d}} \frac{|n(x) - n(y)|^p}{(h + |x - y|)^d} \mathbb{I}_{|x-y|\le 1} \, dx \, dy.$$

This is a sort of extension of usual Sobolev or Besov norms to the case of 0 derivative. Recall for example that for $s > 0$

$$\|n\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^{2d}} \frac{|n(x) - n(y)|^2}{|x - y|^{d+2s}} \, dx \, dy.$$

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However the previous equality fails for $s=0$ and the corresponding norm is stronger than L^2 and is enough to control the high frequencies.

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Proposition

Assume that for a sequence n_k uniformly bounded in L^p

$$\limsup_k |\log h|^{-1} \, \|n_k\|_{h,p}^p \longrightarrow 0 \qquad \text{as } h \to 0,$$

then the sequence n_k is compact in L^p_{loc}.

A quantitative estimate

It is possible to obtain explicit bounds on $\|n_\varepsilon\|_{h,1}$ for the problem we consider.

Proposition

Under all previous assumptions and in particular $a_\varepsilon \in W^{1,p}$ uniformly, one has for a constant C independent of ε and h

$$\|n_\varepsilon\|_{h,1} \leq C\,|\log h|^{1/\bar{p}} + C\,\frac{\varepsilon^2}{h^2} + C\,\|n_\varepsilon^0\|_{h,1} + C\,\|d_\varepsilon\|_{h,1},$$

with $\bar{p} = \min(2, p)$.

Conclusion and perspectives

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- Technical improvements are probably possible: the rate is likely not optimal, the case where $a_\varepsilon \in BV$ ($W^{1,1}$ can be obtained by interpolation).
- Developing good numerical schemes for those models is crucial.
- The condition $\operatorname{div} a_\varepsilon \in L^\infty$ is in principle not necessary for some models. The non linearity lets us keep L^∞ bounds on the density n_ε.

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